System Identification

A Third-year Course for Control and Mechatronics Engineering

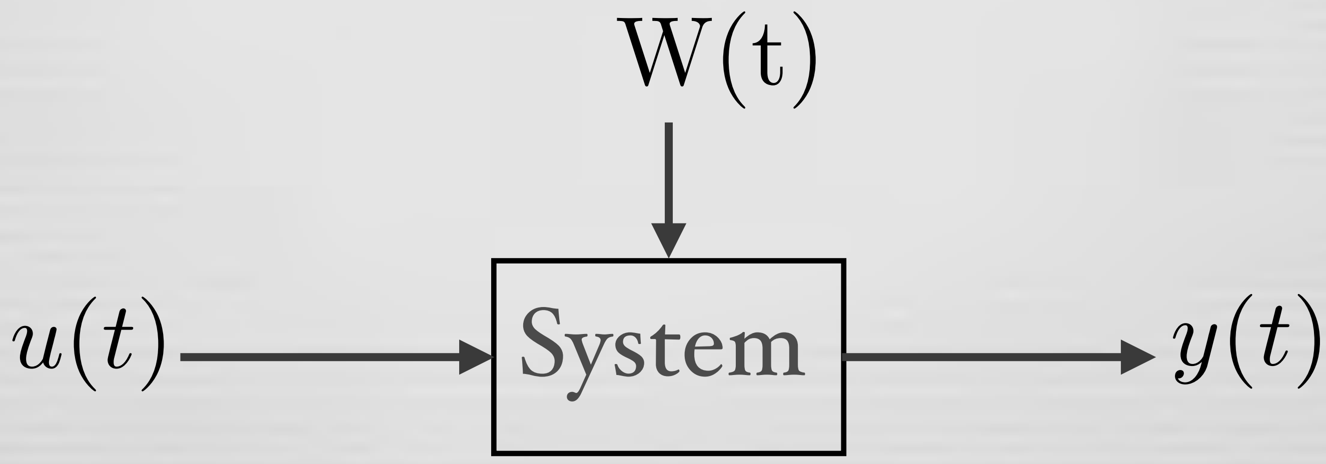
By Dr. Taghreed M. MohammadRidha

Lecture 1

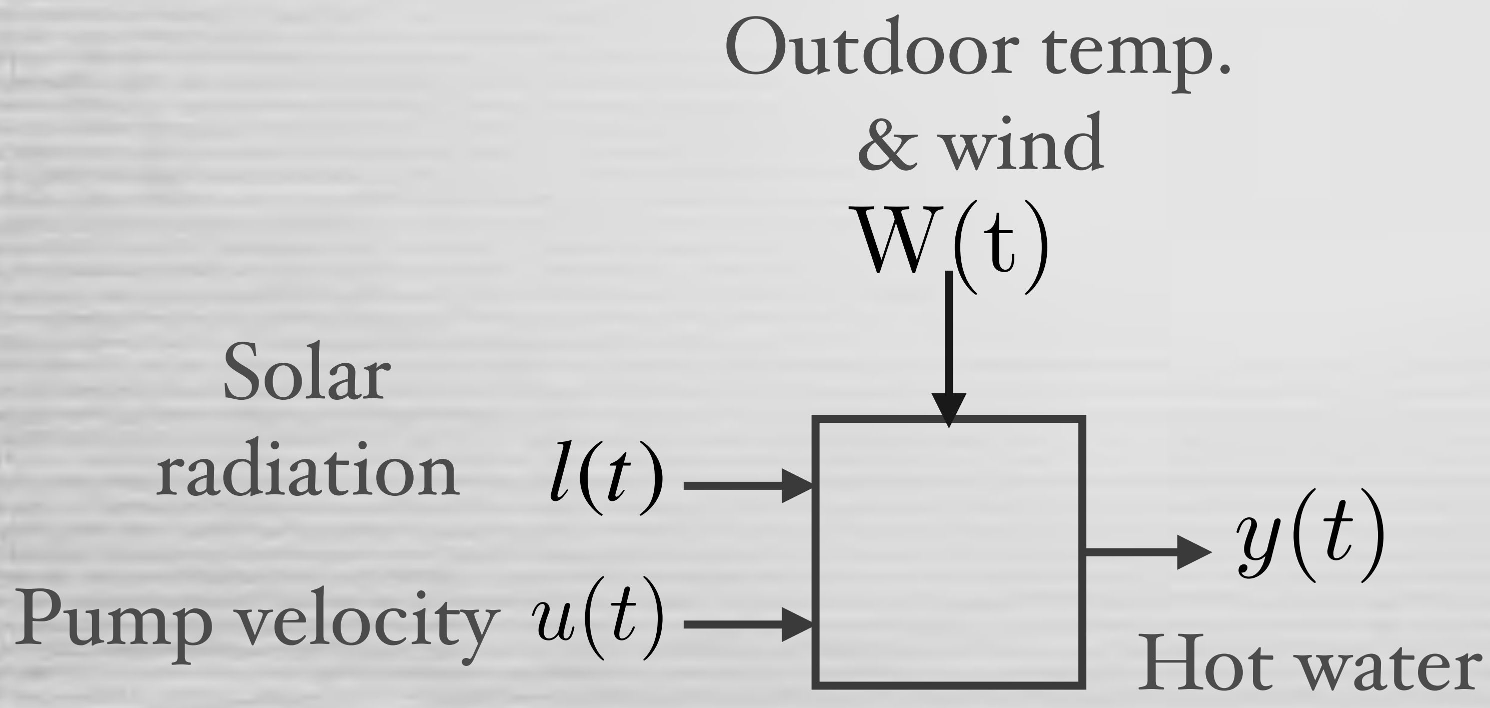
Introduction and Overview

What Is a System?

- **A system** [1]: is an object in which variables of different kinds interact and produce observable signals: *Outputs*.
- Its external signals are either *Inputs* or *Disturbances*.



- Dynamic system: A system with a *memory*, i.e., the input value at time ***t*** will influence the output at future instants.
- *Example:* Solar Water Heater



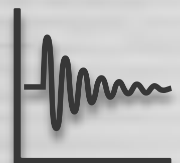
A System MODEL

* A **Model** is a description of a system. The model should capture the essential information of the system.

Where the model is NEEDED?

Where the model is NEEDED? [2]

- In *process design*:  /  leads to difficulties to perform experiments on real process.

- In  *process control:* Short-term behavior of the processed may be needed to be predicted. Used in model-based control design.

- In *plant optimization*, an optimal operating strategy is sought. It may also be used for training the plant personnel.

- In *fault detection*, checking anomalies in process parts. Monitoring physical states (concentration, temp., ... etc.) that are *not available* via measurement.

Types of Models

- *Mental* models do not involve any math formalization, e.g. driving a car.
- *Graphical* models properties are described by numerical tables and/or plots, e.g. step or frequency responses of linear systems.
- *Mathematical* models describe the relationship among system variables in terms of math. expressions, e.g. differential equations.

Building a Model

- A model is constructed from *observed data*.
- Mental model of car-steering dynamics is developed through driving experience.
- Graphical models are developed from measurements.
- Math. models are derived from $\begin{cases} \textit{Modeling} \\ \textit{System Identification} \end{cases}$

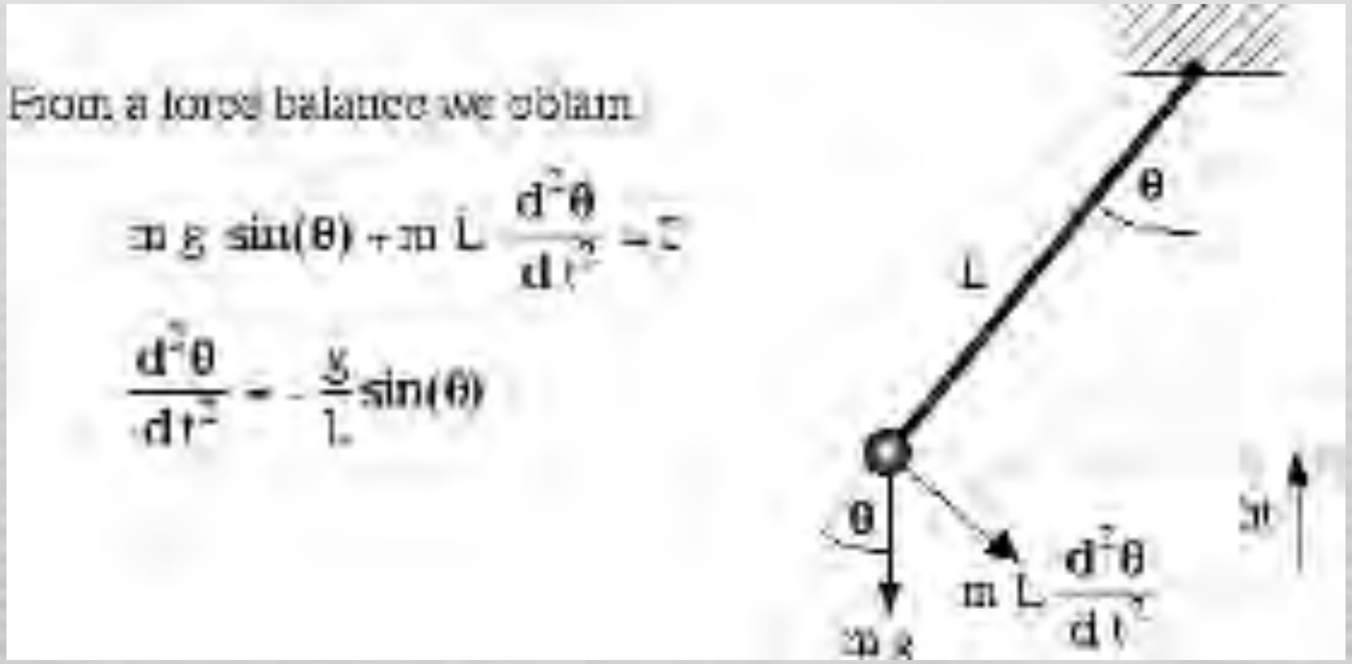
System Identification

- *System Identification* is how to build a system model based on the recorded input and output signals and their data analysis.
- This route to <u>math. as well as to graphical</u> models is based directly on experimentation.
- Generally, our <u>acceptance</u> of a model should be guided by "usefulness" rather than "truth".

Mathematical Model TYPES

1. Linear and nonlinear

- A linear system is a mathematical model of a system based on the use of a **linear** operator. Superposition principle can be applied.
- Nonlinear system: the change of the output is not proportional to the change of the input.



Mathematical Model TYPES

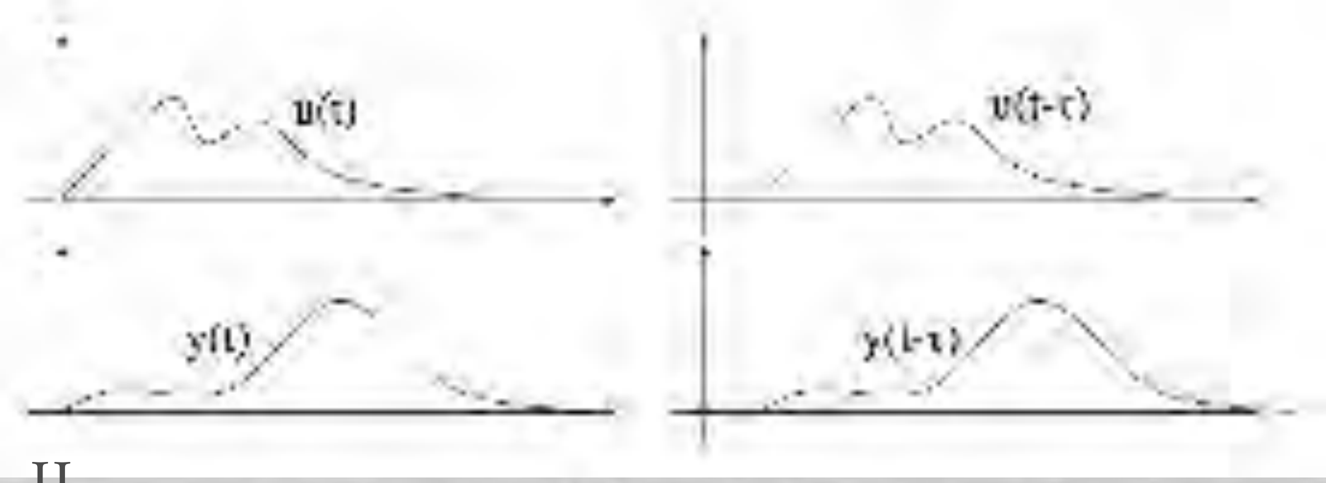
2. Stationary (time invariant) and non-stationary

- Time-varying systems have parameters that vary with time:

$$\frac{dx}{dt} = a(t)x(t)$$

e.g. a rocket mass decreases as fuel is consumed.

- Time-invariant systems:



Mathematical Model TYPES

3. Continues and discrete time

- Continues model: the relationship between continuous signals. Differential equations are often used to describe such a relationship.
- Discrete model: expresses the relationship between the values of the signals at the sampling instants. Such model is typically described by difference equations.

Mathematical Model TYPES

4. Deterministic and stochastic

- Stochastic process: has mainly probabilistic knowledge of the exact state of the system. Uncertainty is present i.e. it's a model for a process that has some kind of randomness.
- Deterministic models (no probabilities): non-parametric models which can be described by (step, frequency,) response. Parametric models which are expressed by differential equations, algebraic equations, T.F's etc.

Mathematical Model TYPES

5. Lumped and distributed

- Distributed parameter model: many physical phenomena are described mathematically by partial differential equations. The events are dispersed over the space variables.
- Lumped models: the events are described by a finite number of changing variables; such models are usually expressed by ordinary differential equations.

Mathematical Model TYPES

6. Static and Dynamic

- Static models: if there are direct, instantaneous links between inputs and outputs, the system is termed static. The input and output are related by algebric equations.
- Dynamic models: inputs and outputs are related by differential equations (which will make the current input affects future outputs also).

Mathematical Model TYPES

6. Single Input and Multi Input

- Identification technique is simplified when the state of the system is affected by one input as compared with a state that is affected by a combination of several inputs.

System Identification

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Lecture 2

- Kinds of models: White box, Grey and Black box.
- Identification vs. Physical modeling
- System Identification Procedure.
- SI Flowchart.
- SI Methods.
- Classical Deterministic Methods: Step Response.

What kinds of models are there?

1-**White box (First-principle)**

* based on physical laws and relationships that cover the system behavior, e.g. mass and energy balances.
* General models: often nonlinear.
* ALL variables & parameters have physical meaning.
* Demands a priori knowledge of the process. Usually Incomplete!!
* Time-consuming.
* May lead to complex models.

What kinds of models are there?

2-Systems Identification (Black Box)

- Use experiments & measurements to deduce a model.
- No or very little prior knowledge is exploited.
- Models are less general.

3-Grey-box models

- Derive model from laws and tune 'some' parameters to data.
- Combines *Analytical models* and *black-box identification*.



White-box, Grey and Black-box models.

A White-box model example: The Simple Pendulum

Force Derivation of a Simple Pendulum

$$F = ma$$

$$F = -mg\sin\theta = ma$$

$$a = -g\sin\theta$$

$$s = \ell\theta$$

$$a = \frac{d^2 s}{dt^2} = \ell\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

Small angle approximation: $\theta \ll 1$

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0.$$

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{\ell}}\,t\right) \qquad \theta_0 \ll 1.$$

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} \qquad \theta_0 \ll 1$$

A Grey-box model example: The Lab.-Scale tank system

- The purpose: water level *y(t)* changes with the inflow generated by the voltage *u(t)*.
- After several experiments, the best linear black-box model:

$$y(t) = a_1 y(t-1) + a_2 u(t-1)$$

A Grey-box model example: The Lab.-Scale tank system

- The fit was not bad BUT the output level was **negative** at certain times!!!
- All tested linear models showed this kind of behavior.
- Combining *Bernoulli's* law: the outflow is proportional to the sqrt(y(t)):

$$y(t) = a_1 y(t-1) + a_2 u(t-1) + a_3 \sqrt{y(t-1)}$$

System Identification Procedure

Input - output data involves four basic ingredients:

- The nature of the input.
- Selection of model structure or determining the order of the linear model.
- Selection of a ID Approach and Parameter Estimation.
- Model validation.

The INPUT

1. The computations can be simplified if a special types of input signals are chosen such as *step, impulse*, Pseudo Random Binary Sequence PRBS, …etc.

2. The input should *excite all the modes* of the system.

3. The choice of the input depends on the type of the input that the process might undergo under normal conditions (operations).

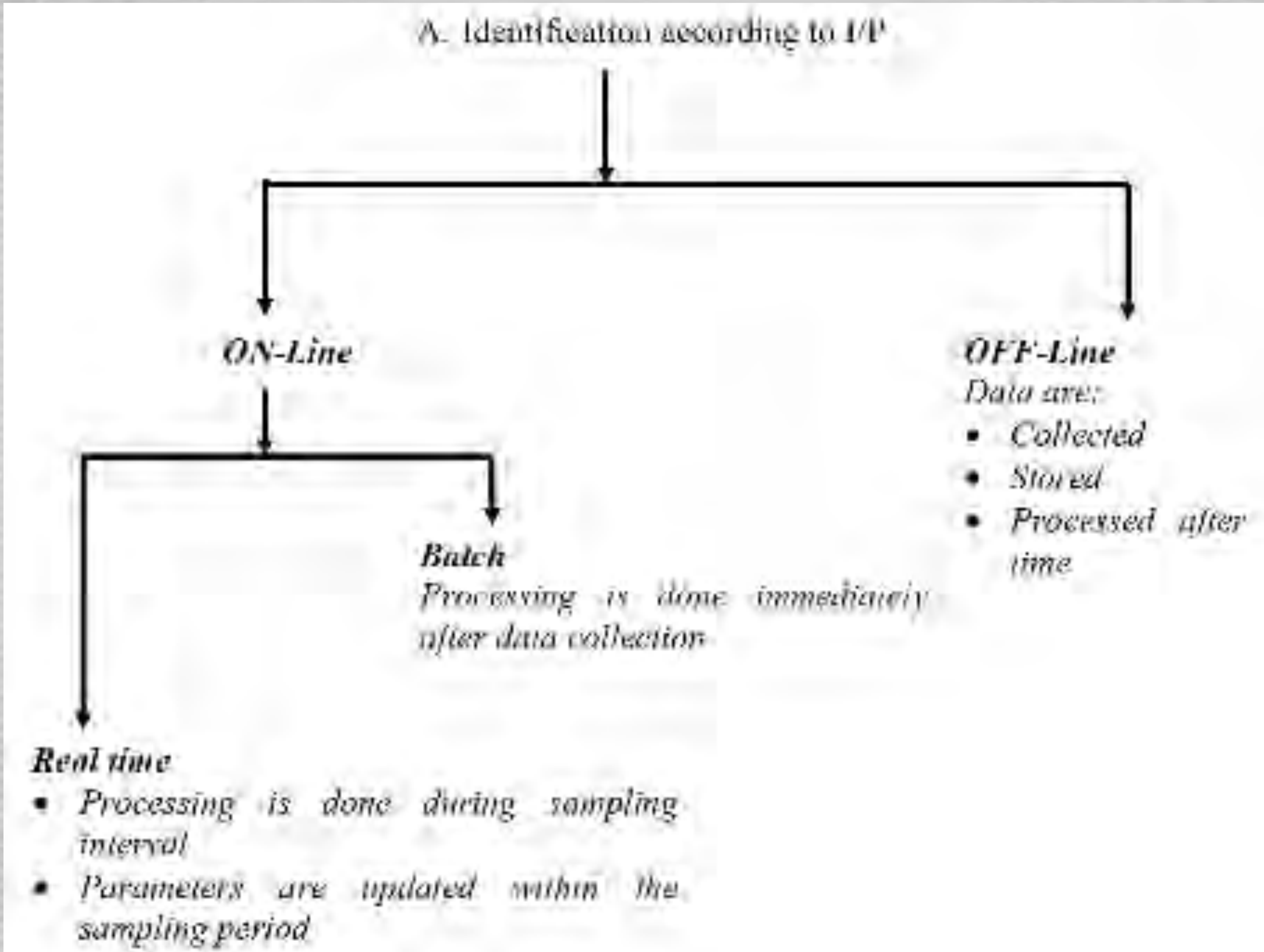
4. The level of the input is chosen such that the process will not drift to nonlinearity or to damage the product of the process.

The Validation

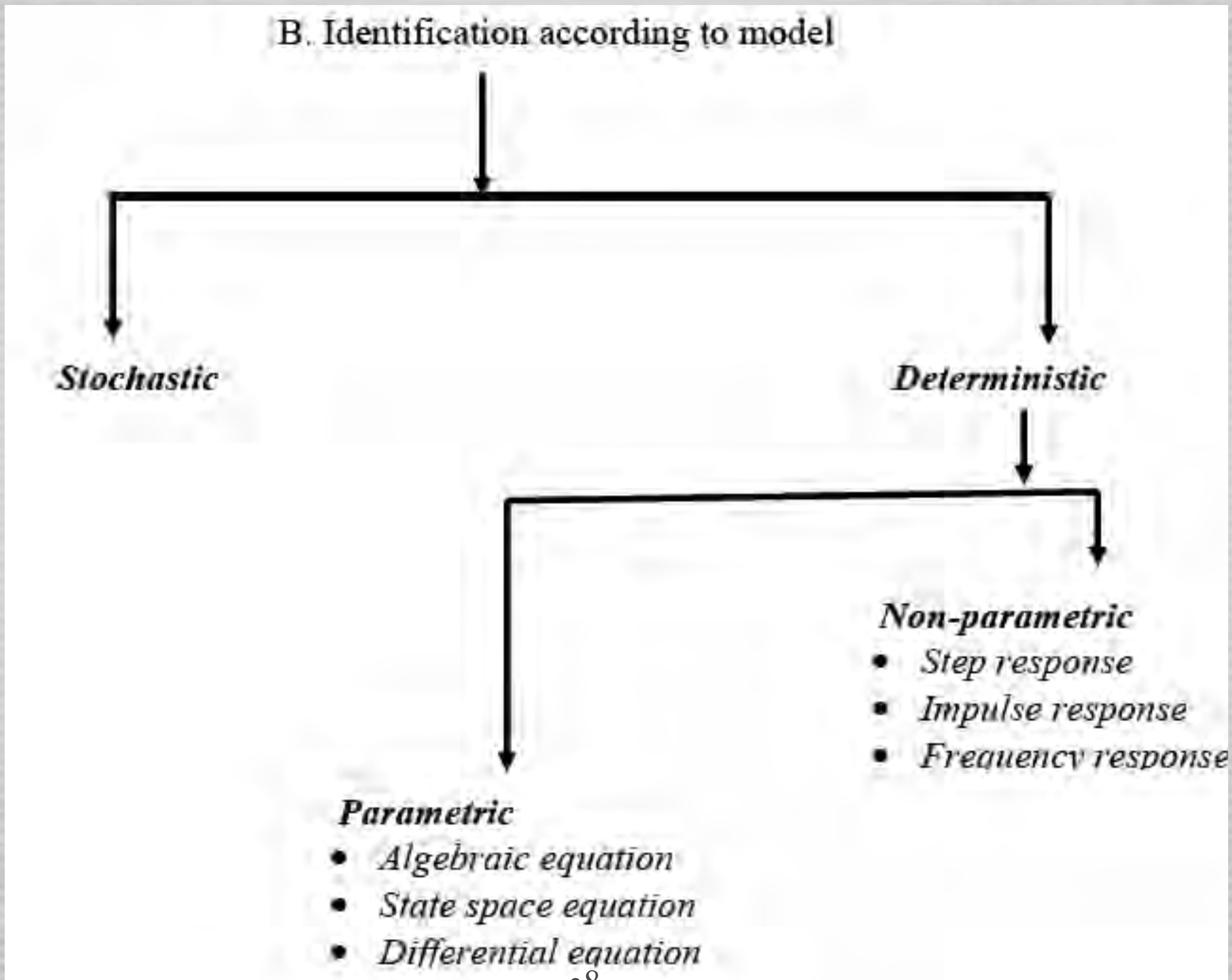
* To validate that the model represents the process or system.
* To see the accuracy, model generalization abilities.
* Cross-validation tests: Difference between the simulated and measured output.
* Process Prior knowledge & statistical tests involving confidence limits are used to validate the model.

However, it must recognize that this objective of proving the model is correct can only be approached and not achieved.

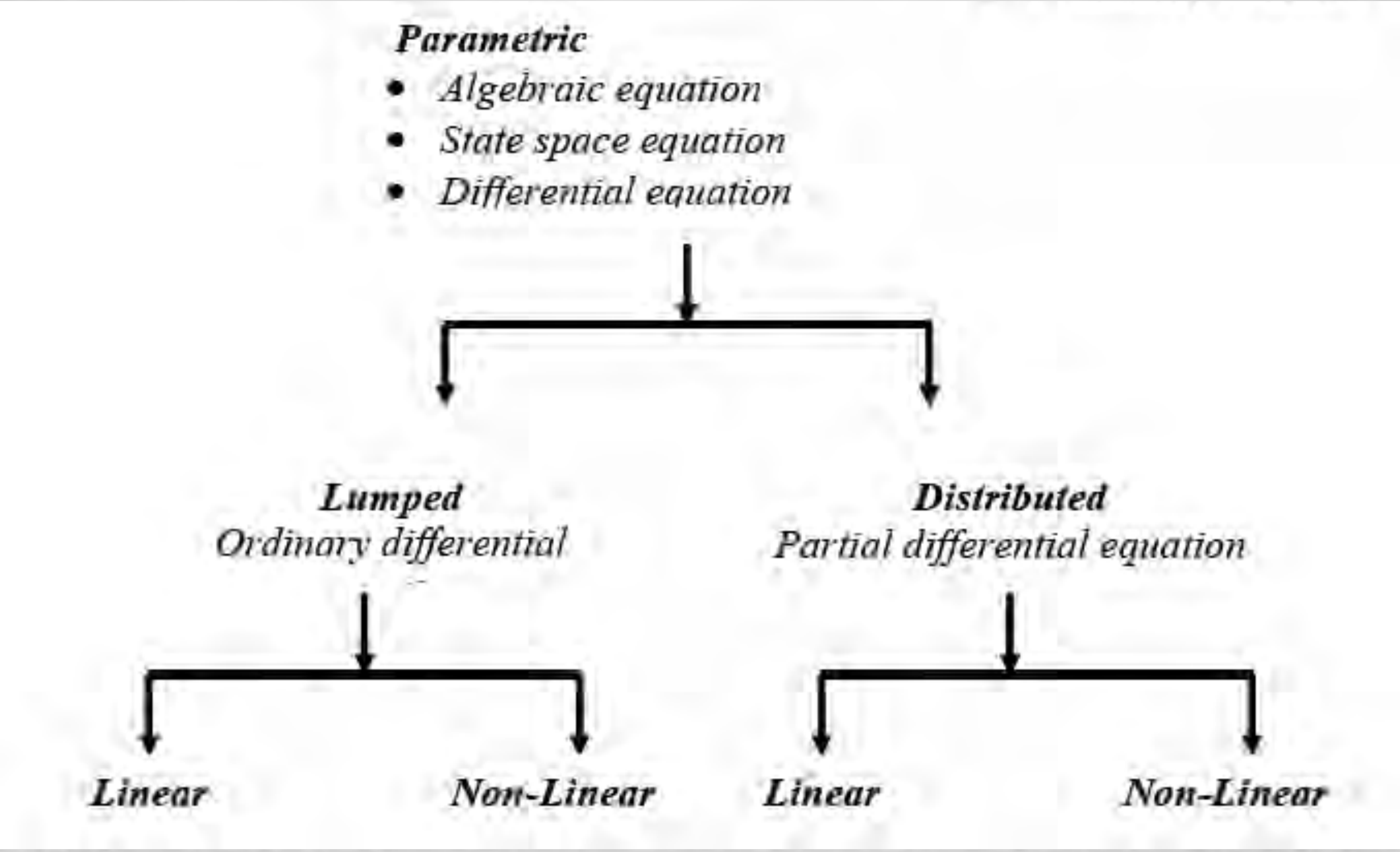
Classification of Identification Methods



Classification of Identification Methods



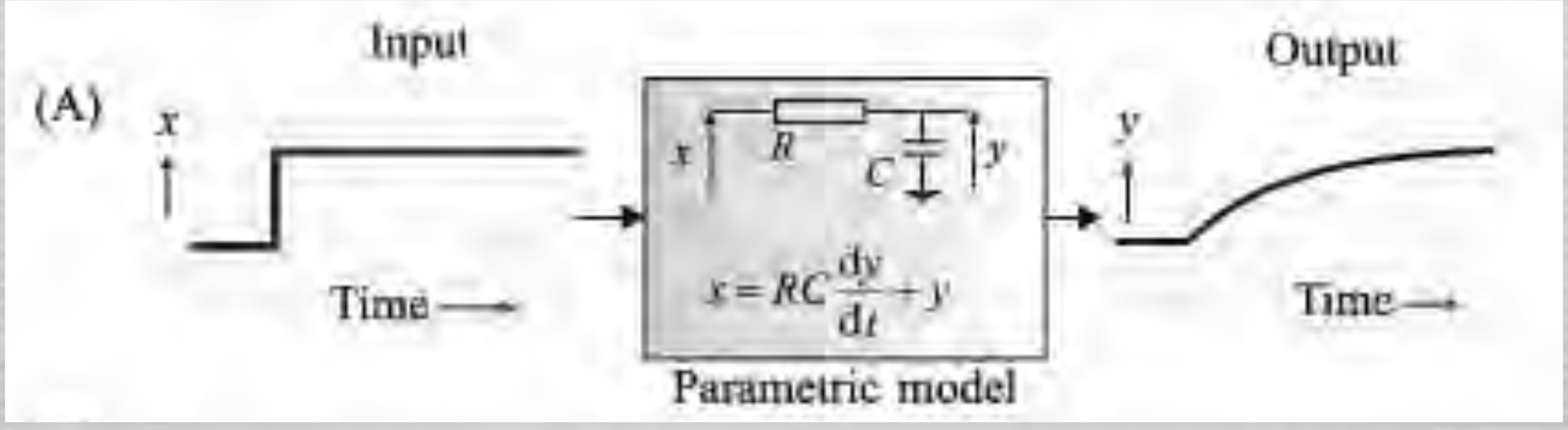
Identification according to a Model



Parametric Model

- The results are values of the parameters in the model.
- These may provide better accuracy (more information), but are often computationally more demanding.

(A) Example of a parametric model of a dynamical linear system (a low-pass filter) and its input & output (x and y).



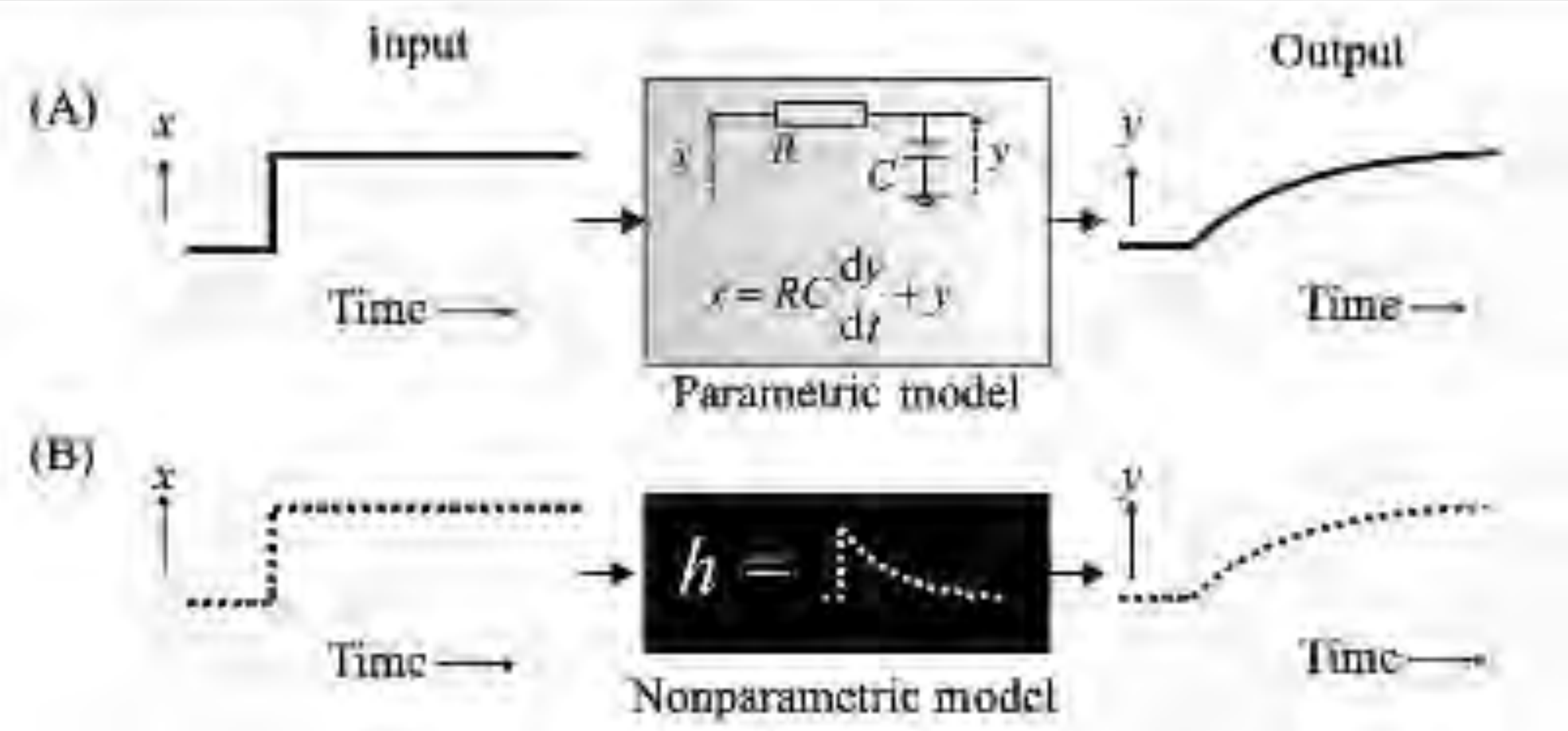
Non-parametric model

- Have a large number of parameters.
- These parameters do not necessarily have a physical interpretation.
- Generally, a nonparametric model is generated from a procedure in which we relate a system's input x(t) and output y(t).
- The results are (only) curves, tables, etc.
- These methods are simple to apply.
- They give basic information about e.g. *time delay, and time constants*

of the system.

- Example: the characterization of an LTI dynamical system with its (sampled) unit impulse response(UIR). The operator in this case would be convolution.

Parametric vs. Non-parametric model

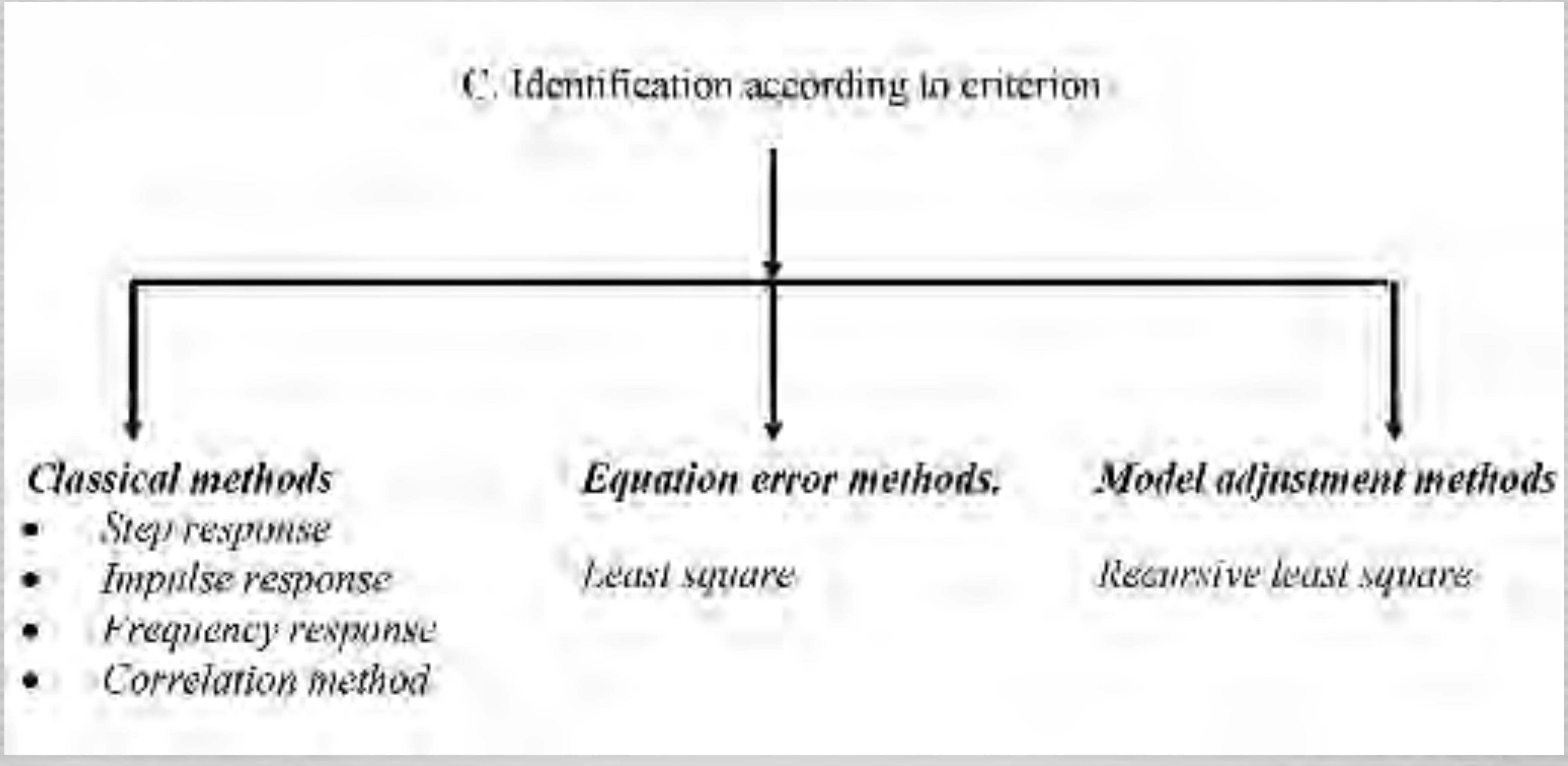


(B) The <u>black box</u>, nonparametric equivalent of the same system is the white curve representing the (sampled) unit impulse response (UIR).

- Convolution of the input time series *x(t)* with the system's UIR *h(t)* generates the system's output time series *y(t)*:

$$y(t) = h(t) \otimes x(t)$$

Classification of Identification Methods



The basic steps of SI

The identification process amounts to repeatedly selecting a model structure, computing the best model in the structure, and evaluating this model's properties to see if they are satisfactory. The cycle can be itemized as follows:

1. Design an experiment and collect input-output data from the process to be identified.

2. Examine the data. Polish it so as to remove trends and outliers, and select useful portions of the original data. Possibly apply filtering to enhance important frequency ranges.

The basic steps of SI

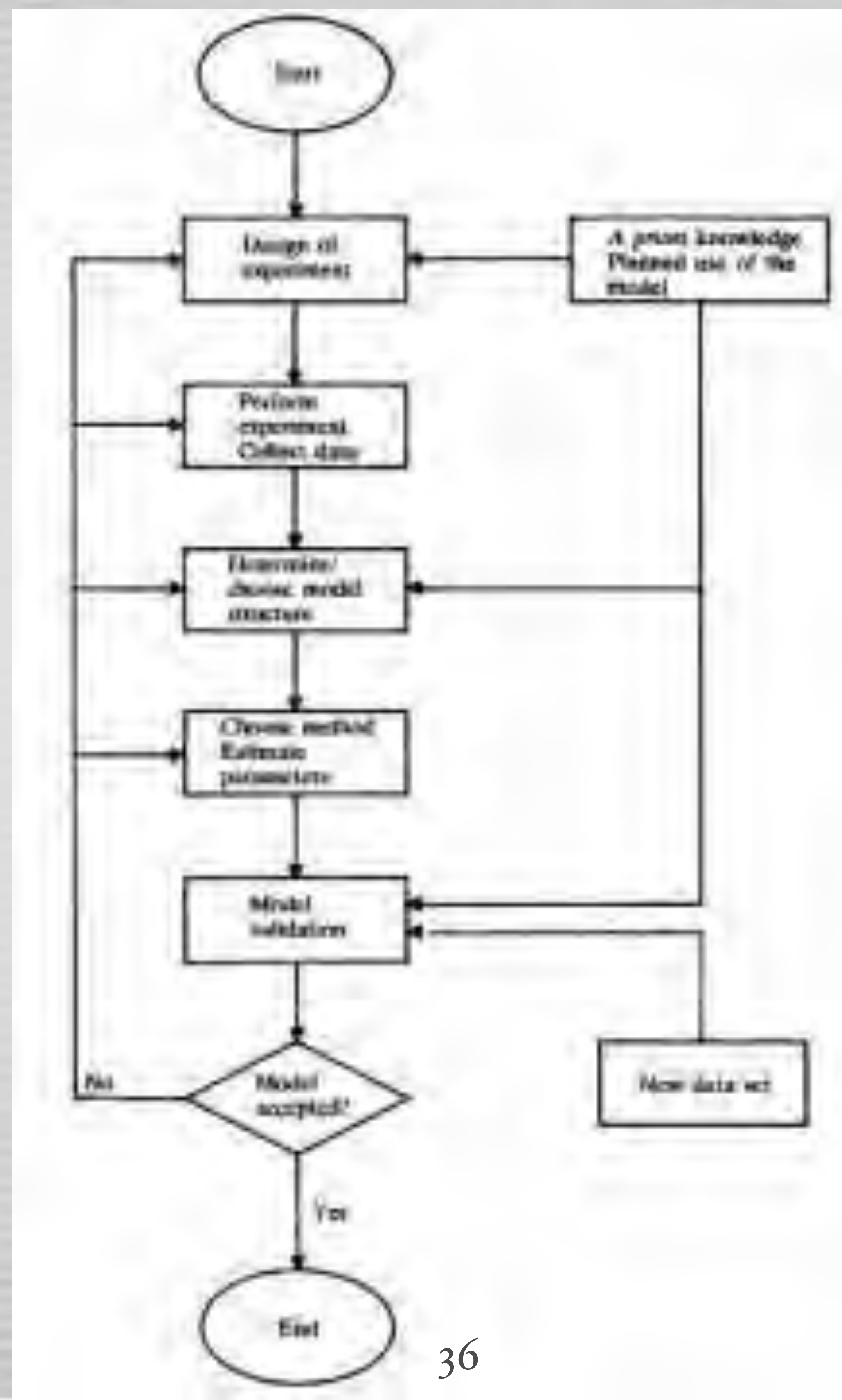
3. Select and define a model structure according to the input-output data.

4. Compute the best model in the model structure according to the input-output data and a given criterion of fit.

5. Examine the obtained model's properties

6. If the model is good enough, then stop; otherwise go back to Step 3 to try another model set. Possibly also try other estimation methods (Step 4) or work further on the input-output data (Steps 1 and 2).

System Identification Flow Chart



Identification from Step Responses

A classical Method

Identification from Step Responses

* Provide information about an approximate process gain, dominant time constant, and time delay.
* The input signal used is a *step change of one of the process inputs* when all other inputs are held *constant*.
* It is necessary that the controlled process is in a steady state before the step change.
* The measured process response is a real step response that needs to be further *normalized for unit step change and for zero initial conditions*.
* Taking *several step* responses and calculating *the average* from them may help to diminish the effect of random noise.

Identification from Step Responses

* When exposed to a sudden change in the input, the system will initially have undesirable output period known as *transient response*.

* The *steady state response* of the system is the response *after the transient response has ended*.

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First-order System [3]

- Consider a first order approximation of an identified process:

$$G(s) = \frac{Z}{Ts + 1} e^{-T_d s}$$

- where Z is the process gain, T time constant, and T_d time delay that need to be determined.

- The step response corresponding to $G(s)$ can be obtained via the inverse Laplace transform of the output as:

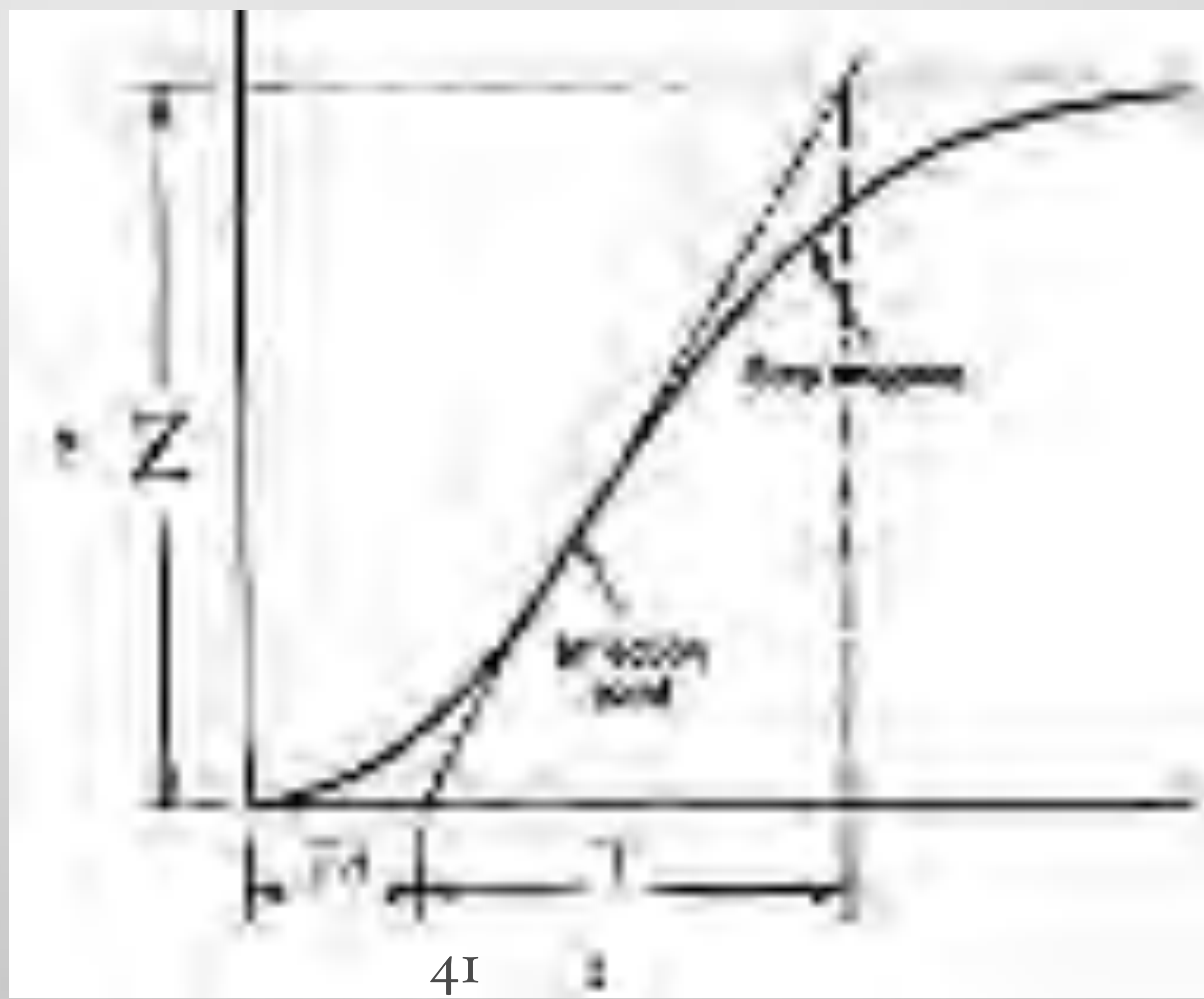
$$y(t) = \begin{cases} 0 & t < T_d \\ Z\left(1 - e^{-\frac{t - T_d}{T}}\right) & t \geq T_d \end{cases}$$

Prove

FOS Model from Step Response

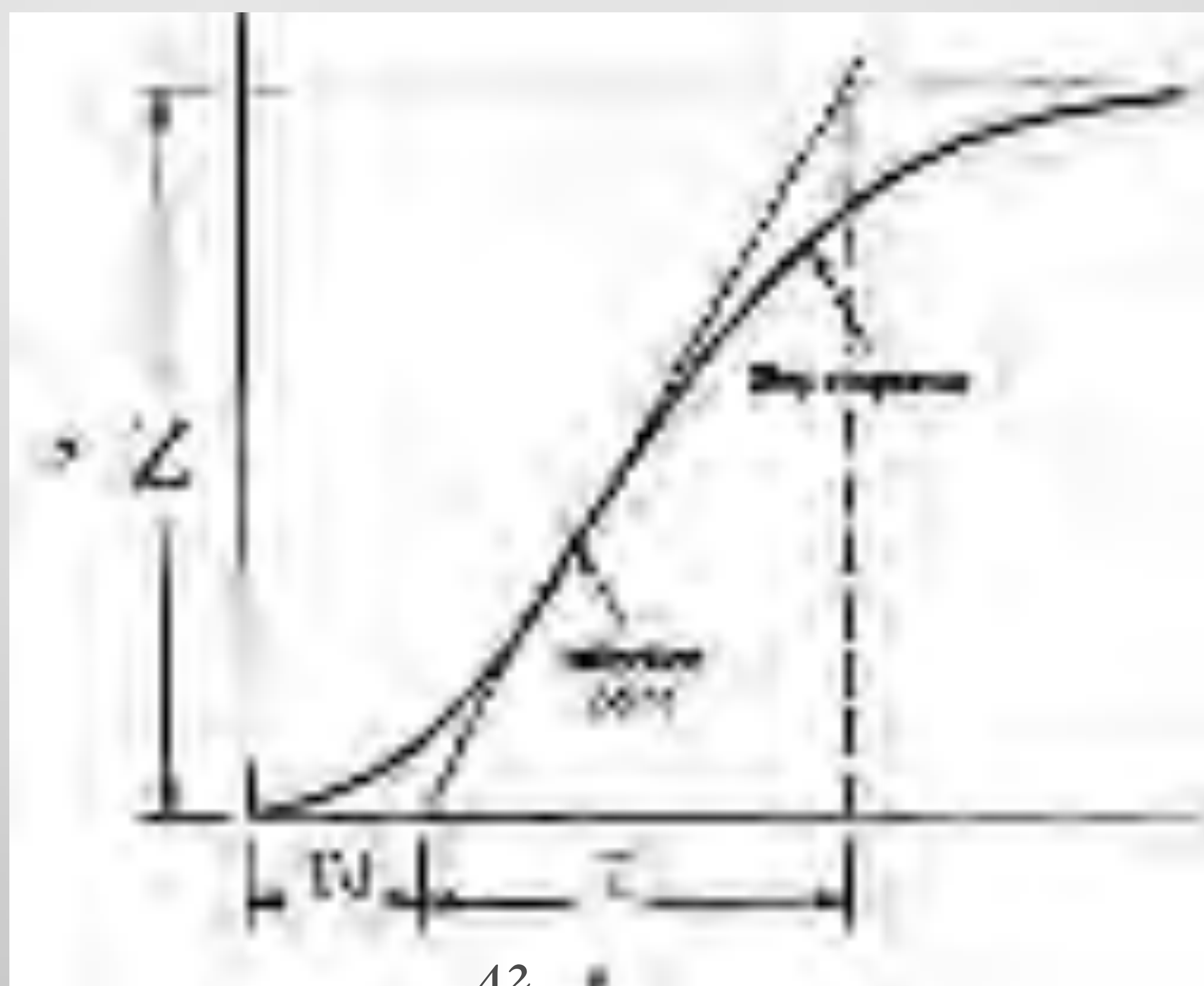
Graph. method: *Slope-intercept method*

- First, a slope is drawn through *the inflection point* of the process reaction curve.
- Then T and Td are determined by inspection.

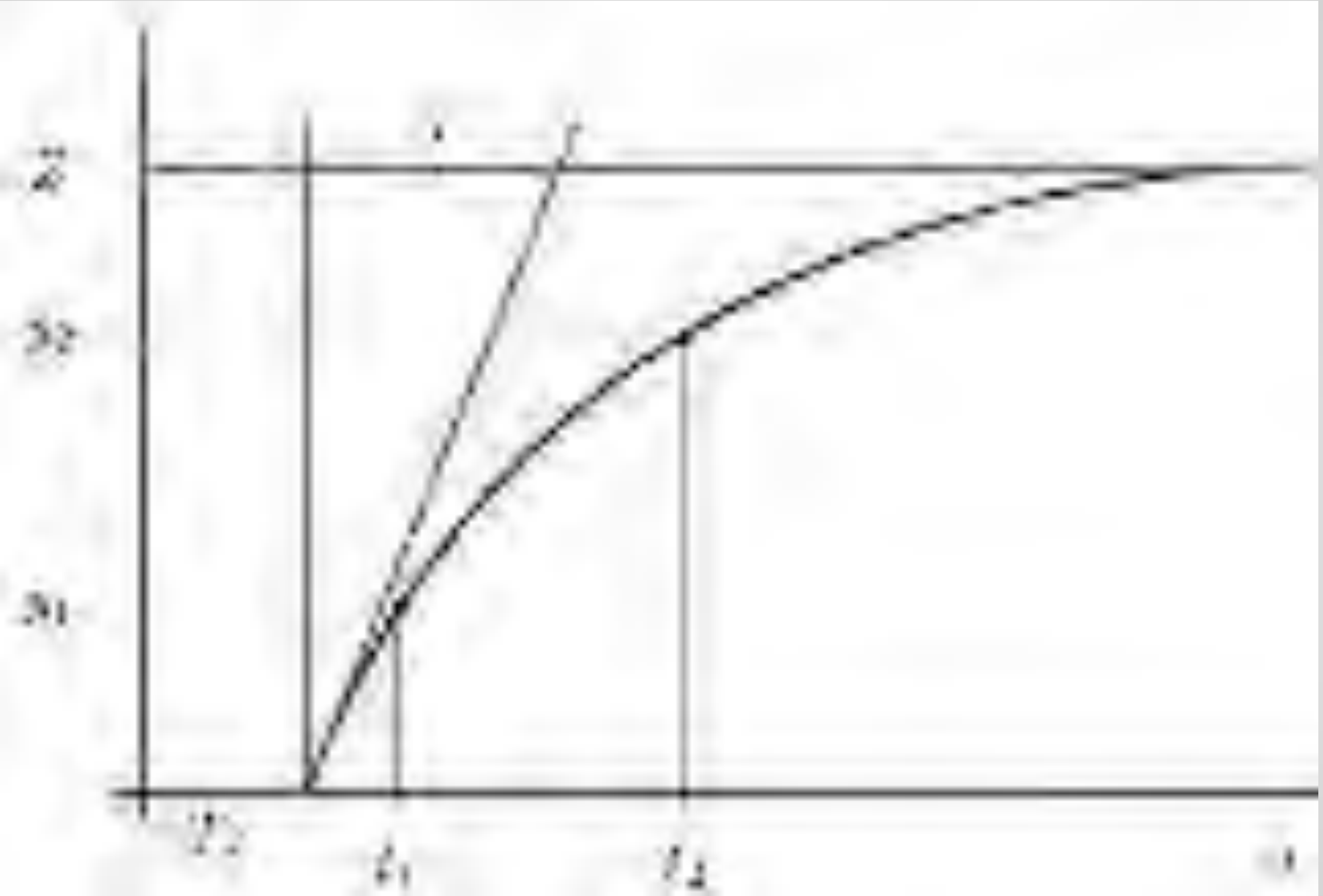


FOS Model from Step Response

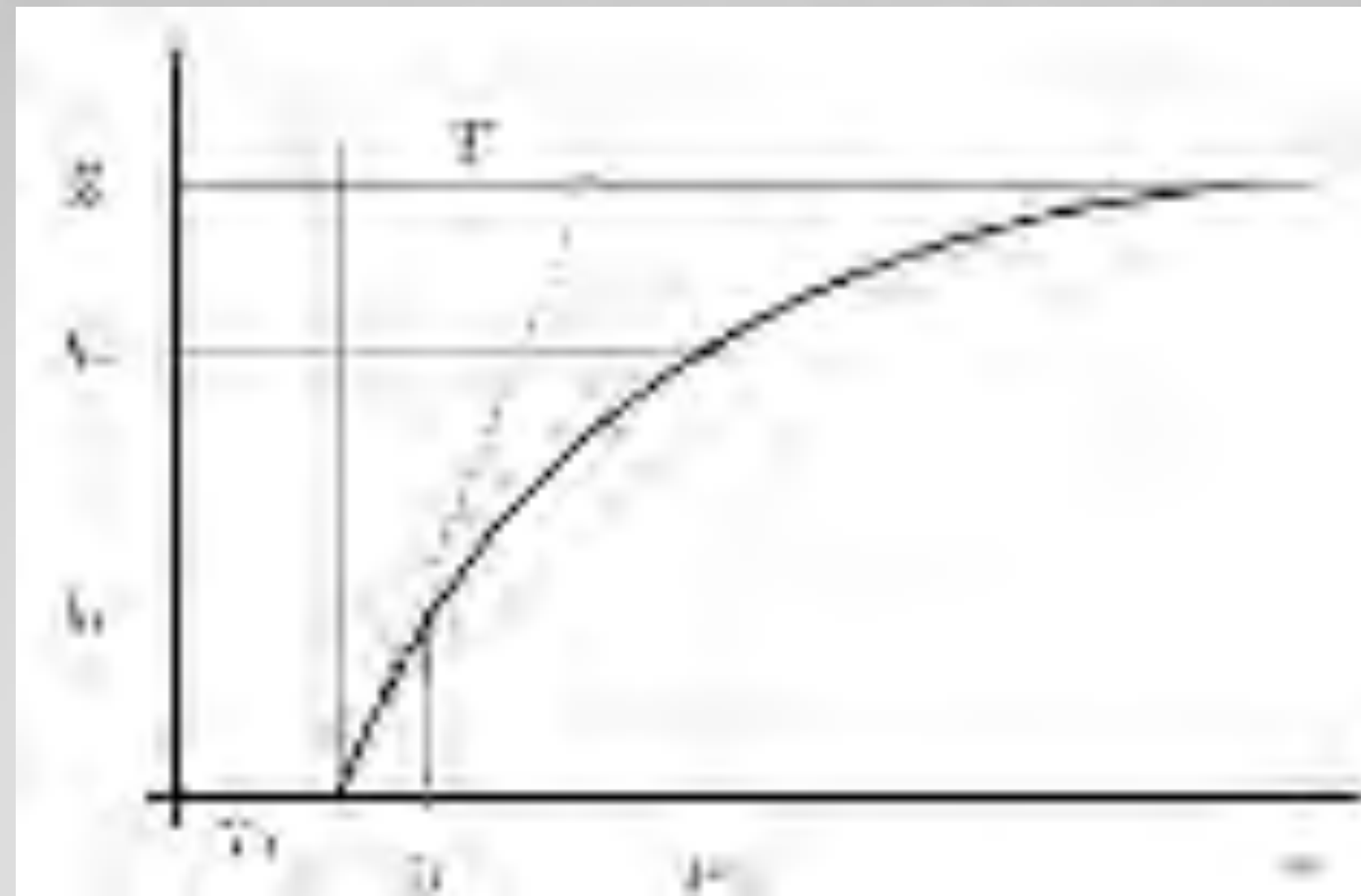
- This method suffers from using only <u>one point</u> to estimate T.
- difficult to find the inflection point due to e.g. noise, computer display,...etc
- *Several points* may provide better estimate.
- <u>T</u> can also be obtained from the step response as the time when the output reaches <u>63%</u> from its new steady state.



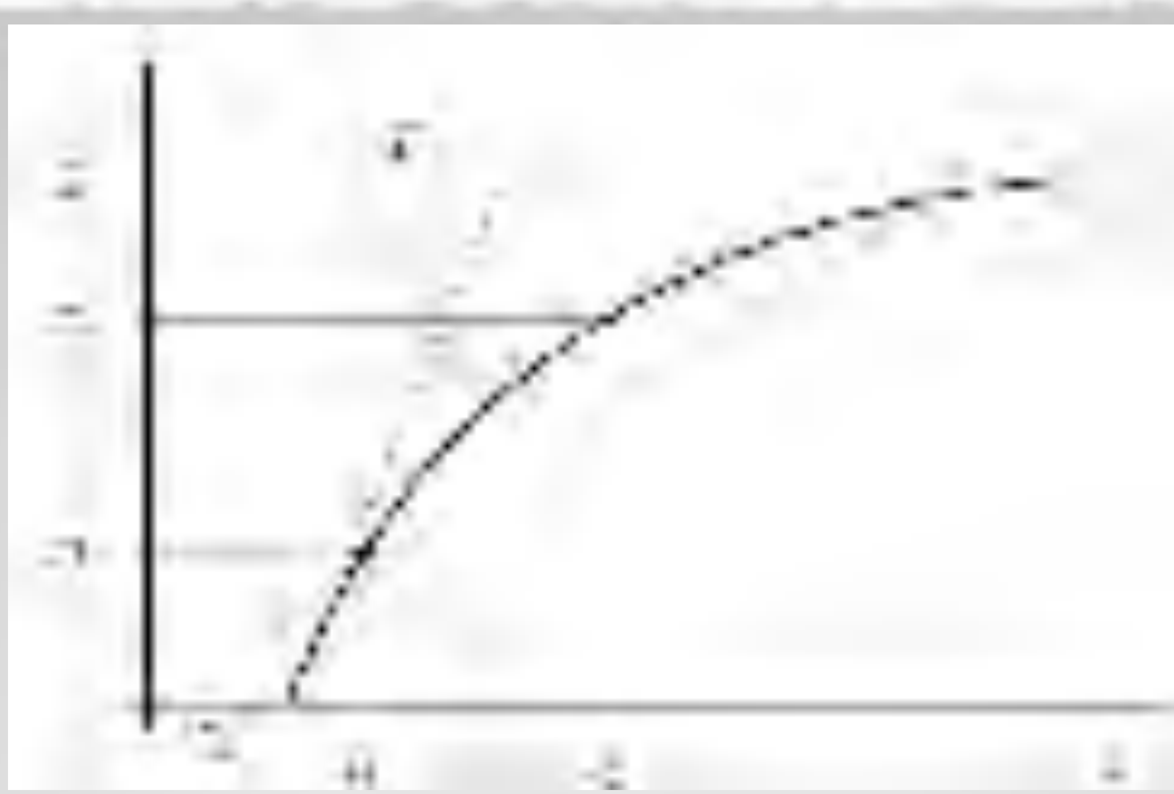
Step Response of a First-order system



$$y(t) = \begin{cases} 0 & t < T_d \\ K\left(1 - e^{-\frac{t - T_d}{T}}\right) & t \geq T_d \end{cases}$$



- We assume *the normalized* step response.
- To normalize: $y(t)/\Delta u$, Δu is the step change.
- The process static gain is given as the new steady-state output $Z = y(\infty)$ ....(Prove!)



- If we assume that two points *t1, y1* and *t2, y2* from the step response are known then:

$$y_1 = K\left(1 - e^{-\frac{t_1 - T_d}{T}}\right)$$

$$y_2 = K\left(1 - e^{-\frac{t_2 - T_d}{T}}\right)$$

After some manipulation:

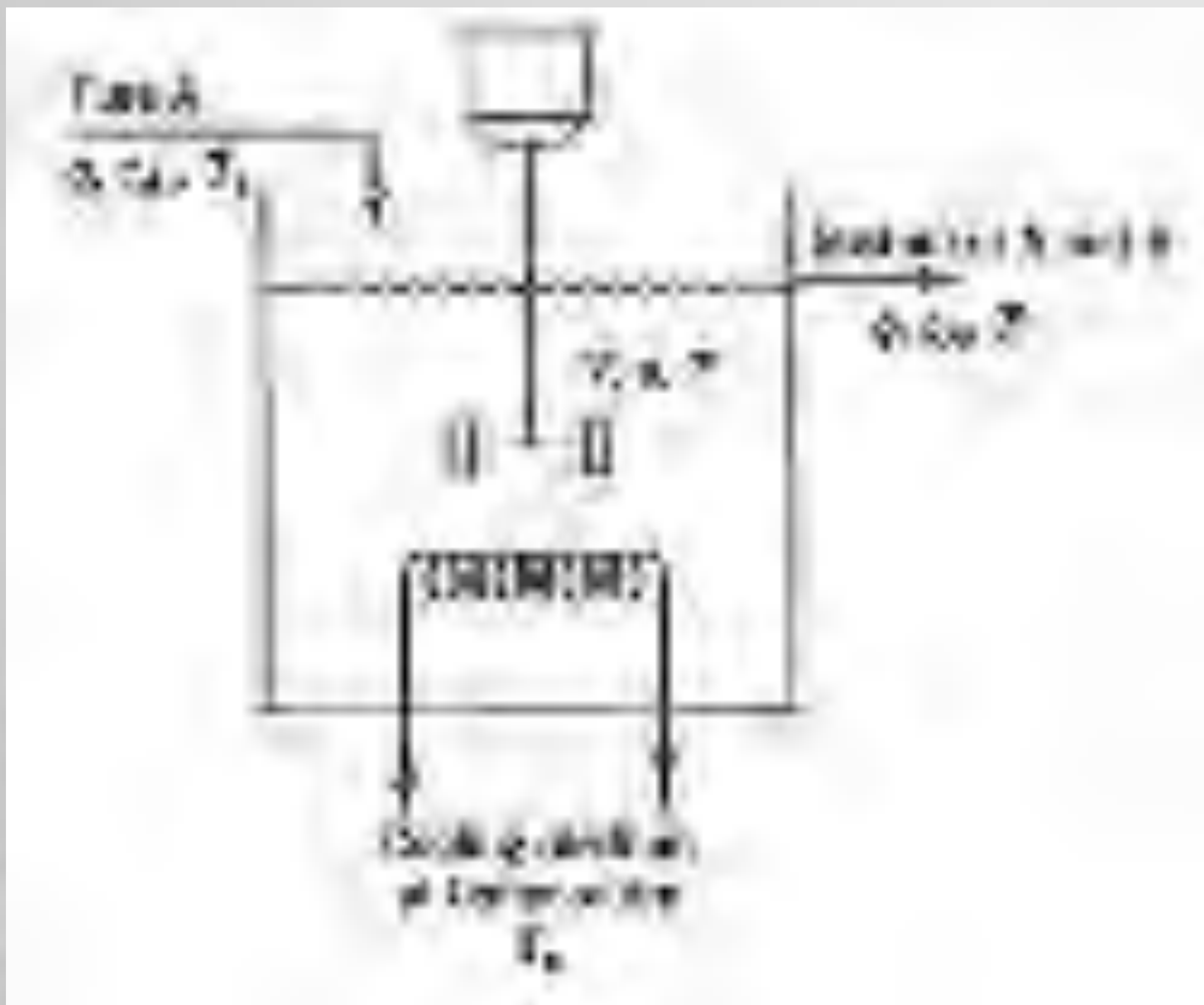
[illegible]

Prove

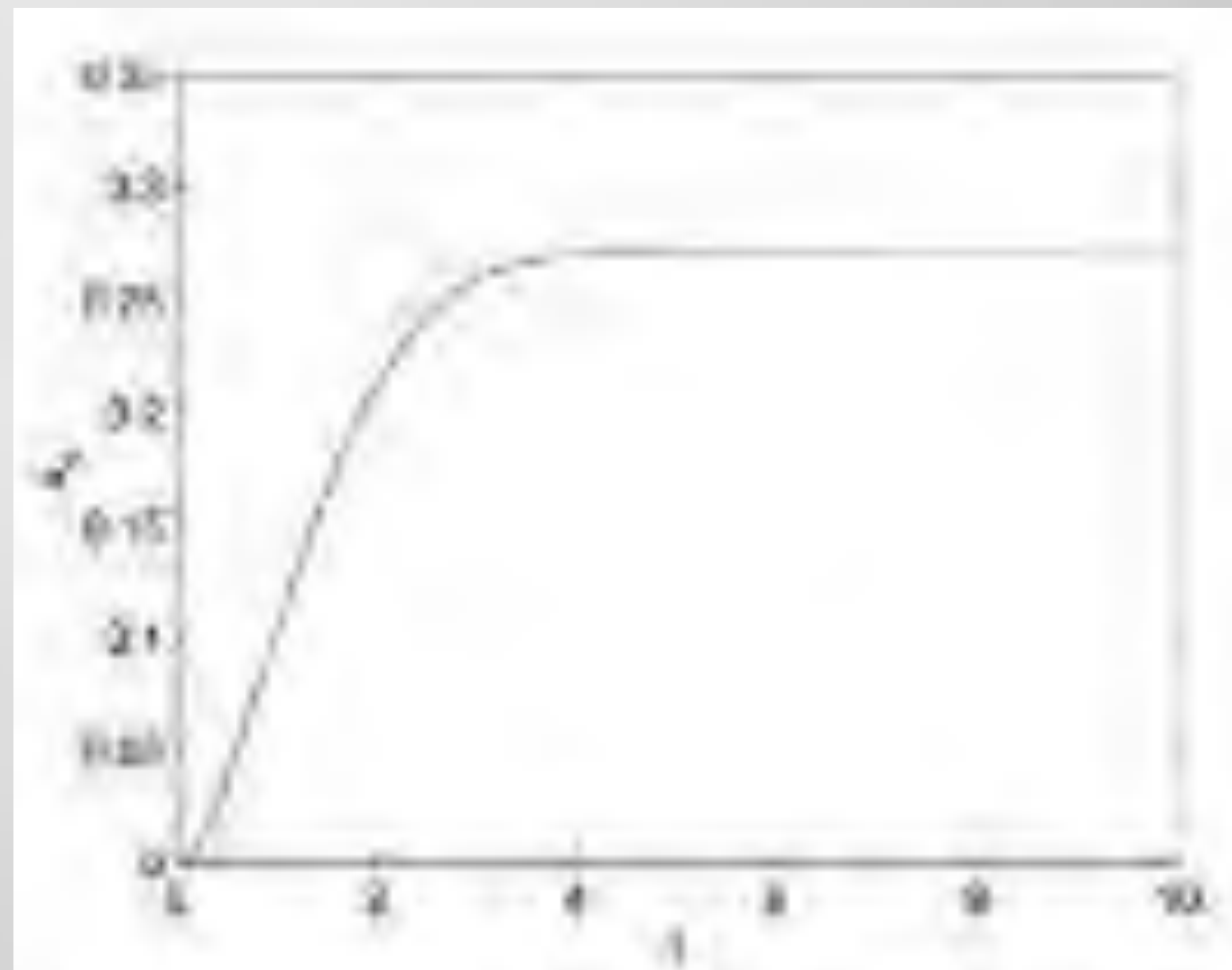
Example: CSTR Model

Example: SI of First-order System

- Consider step response of dimensionless deviation output concentration *x1* in a CSTR to step change of Δqc = 10

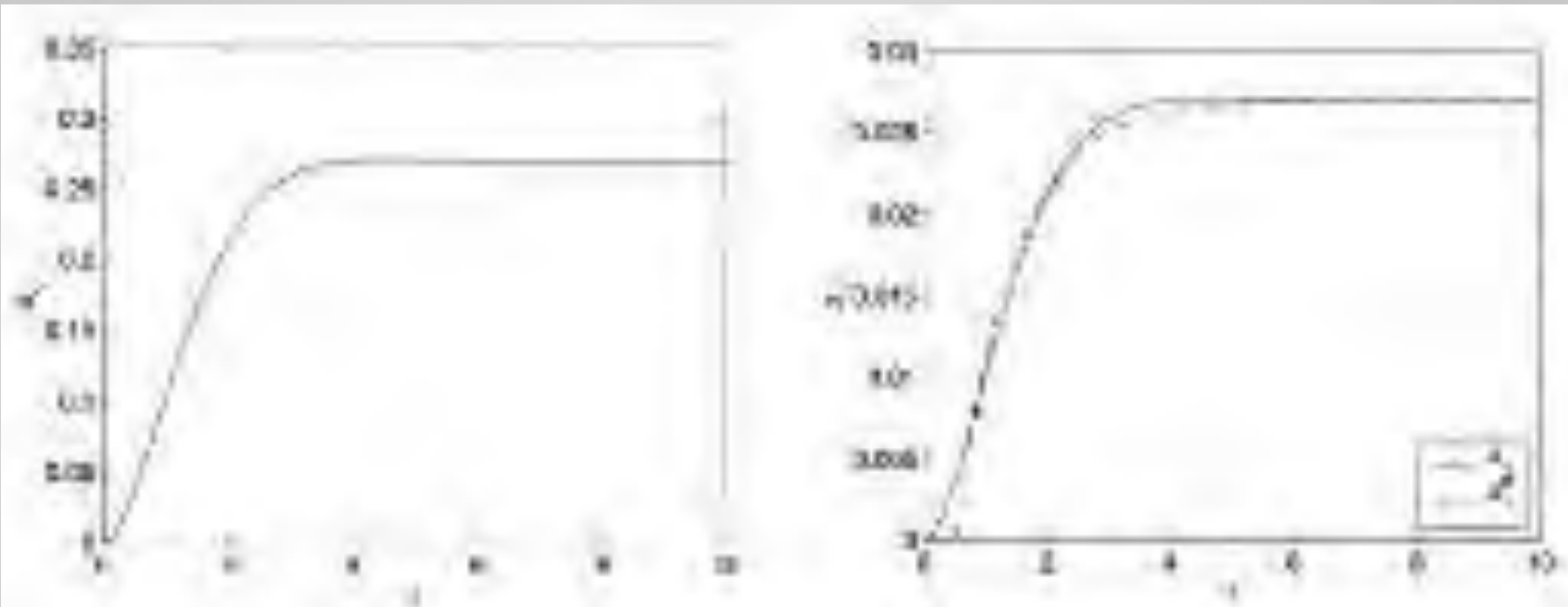


Continues Stirred-Tank Reactor (CSTR)



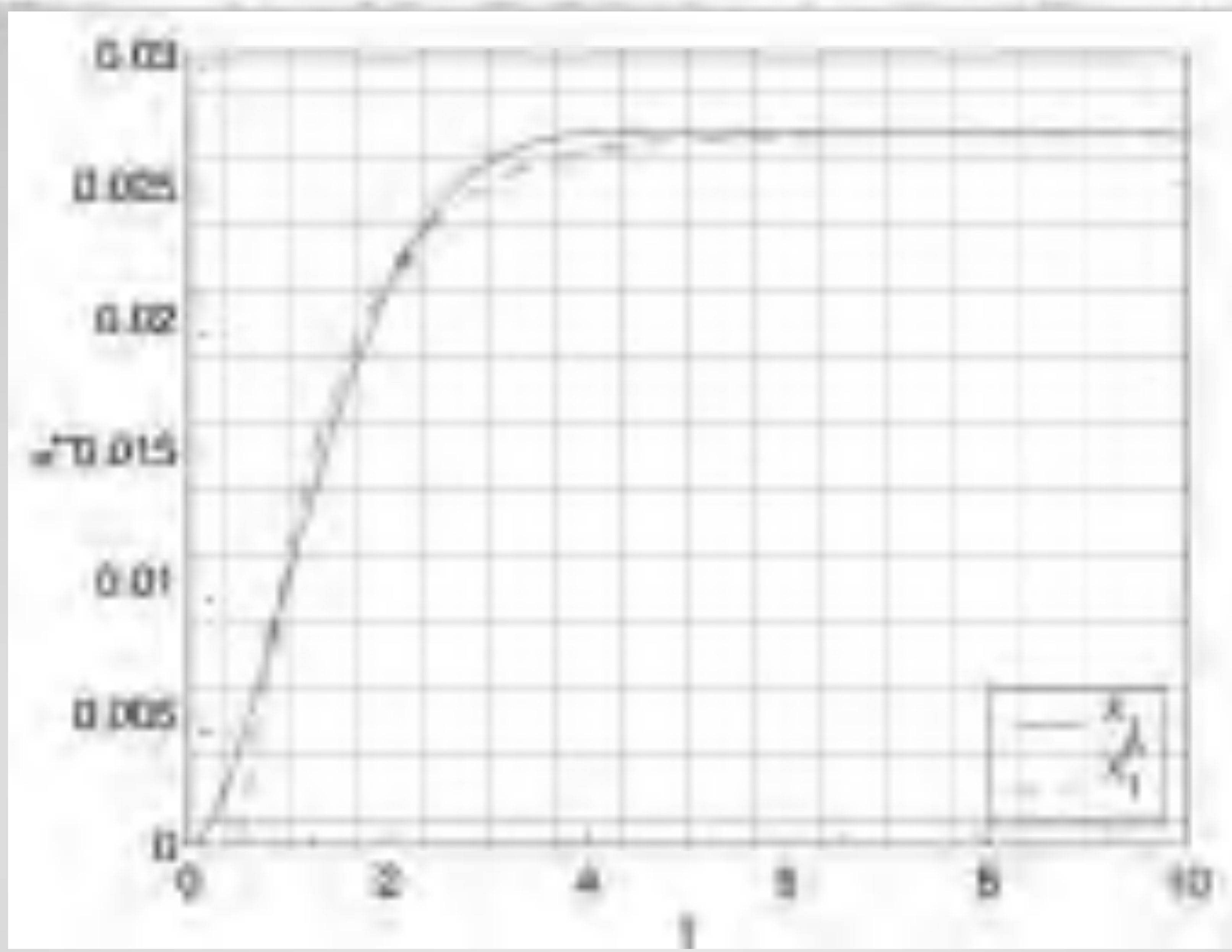
Measured step response of a chemical reactor using the input change Δu = 10

Example: SI of First-order System



- Consider step response of dimensionless deviation output concentration *x1* in a CSTR to step change of Δqc = 10

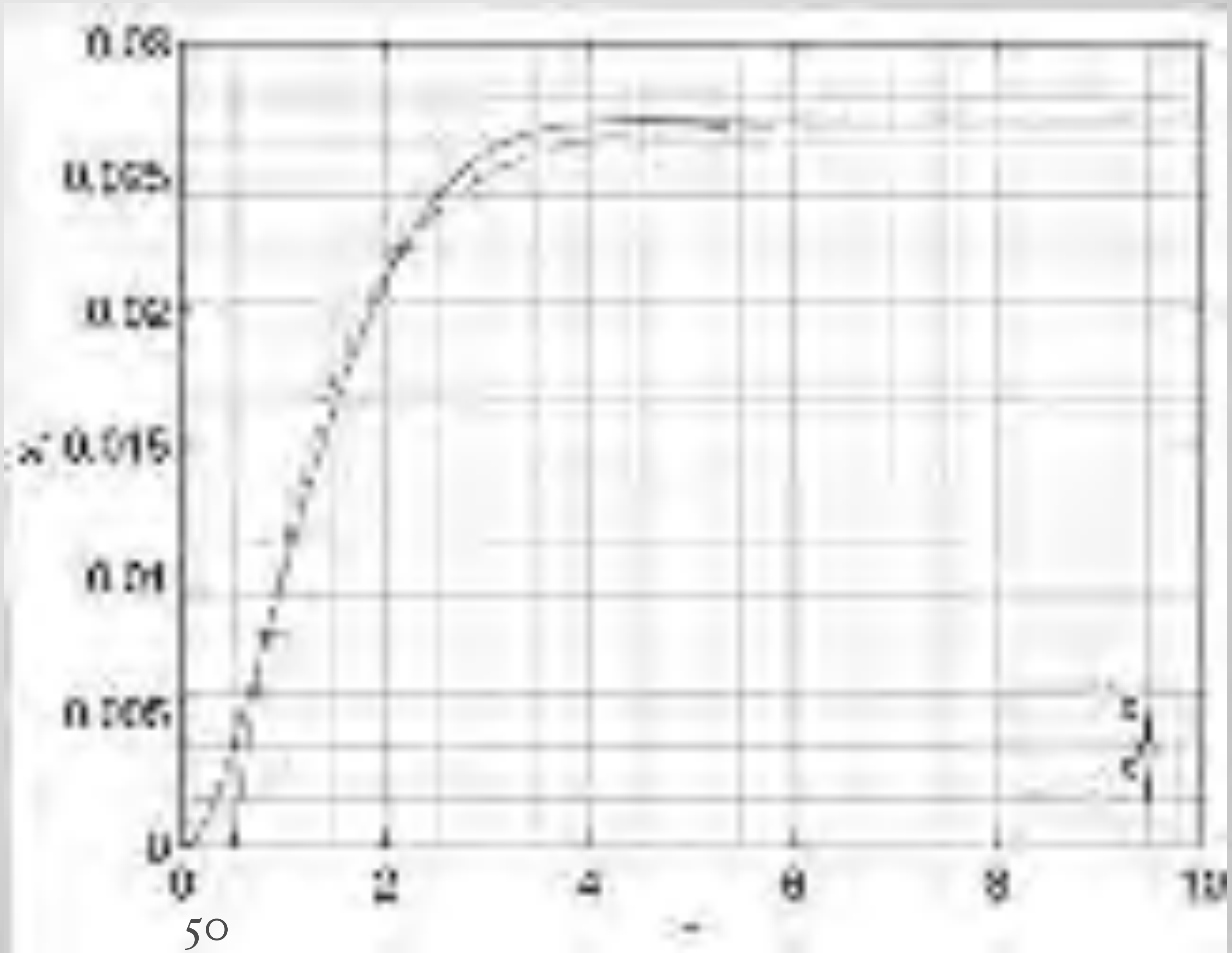
- To obtain the normalized step response, the original one was divided by the step change value.



- Two points were chosen: [0.85; 0.0082] and [2.18; 0.0224].
- The process gain was obtained as Z = 0.027 from $y(\infty)$.
- From the points t_1, y_1 and t_2, y_2, the time constant T = 0.94 and the time delay Td = 0.51.

- The approximated step response is shown in the same figure by a dashed line.
- Both curves coincide at the measured points.
- However, there are significant discrepancies elsewhere.
- This procedure serves only for a crude estimate of process parameters.

$$G(s) = \frac{0.027}{0.94s + 1} e^{-0.5s}$$



FOS Model from Step Response

Sundaresan and Krishnaswamy's Method

Sundaresan and Krishnaswamy's Method

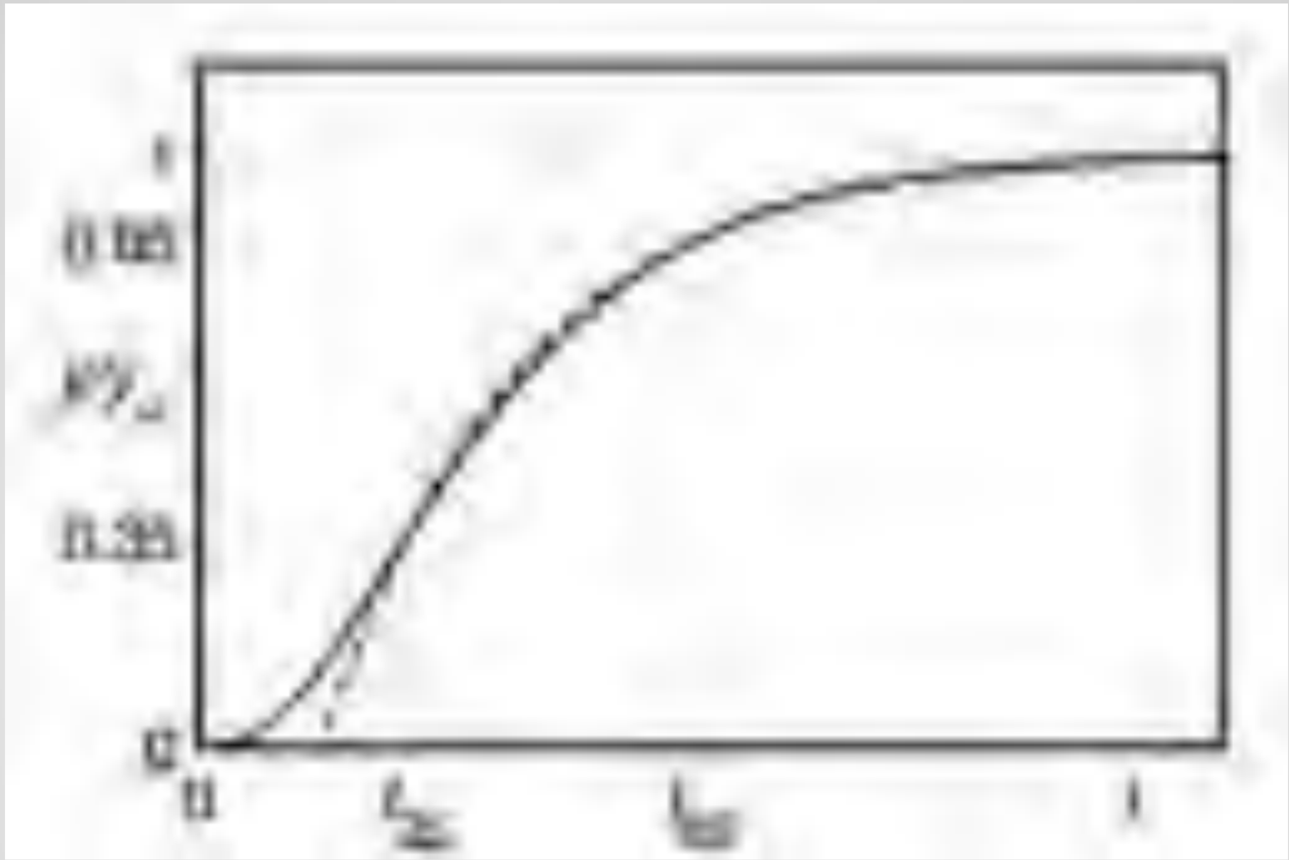
- They proposed that two times, t_{35} and t_{85}, be estimated from a step response curve, corresponding to the 35.3% and 85.3% response times, respectively.
- T and Td are then estimated from the following equations:

$$T_d = 1.3t_{35} - 0.29t_{85}$$

$$T = 0.67(t_{85} - t_{35})$$

- Z can be calculated by the ratio of total steady-state change in y and the size of step change of u.

Sundaresan and Krishnaswamy's Method



$$T_d = 1.3t_{35} - 0.29t_{85}$$

$$T = 0.67(t_{85} - t_{35})$$

$$Z = y_\infty$$

Estimating Second-order Model Parameters Using Graphical Analysis

- In general, a better approximation to an experimental step response can be obtained by fitting a *second-order* model to the data.

Second-order LTI Model [4]

$$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

where K is the system gain, ζ is called relative damping and ω_n is a constant called undamped natural frequency. Sometimes the following form is also used

$$G(s)=\frac{K}{\tau^2 s^2+2\zeta\tau s+1}$$

where $\tau = 1/\omega_n$. There is no universally accepted term for τ, both natural period and second-order time constant are used.

The system is said to be underdamped if $0 \le \zeta < 1$, critically damped if $\zeta = 1$ and overdamped if $\zeta > 1$. If $\zeta < 0$ the system is unstable.

Fitting Second -order Models

- A 2nd order system TF is written often as

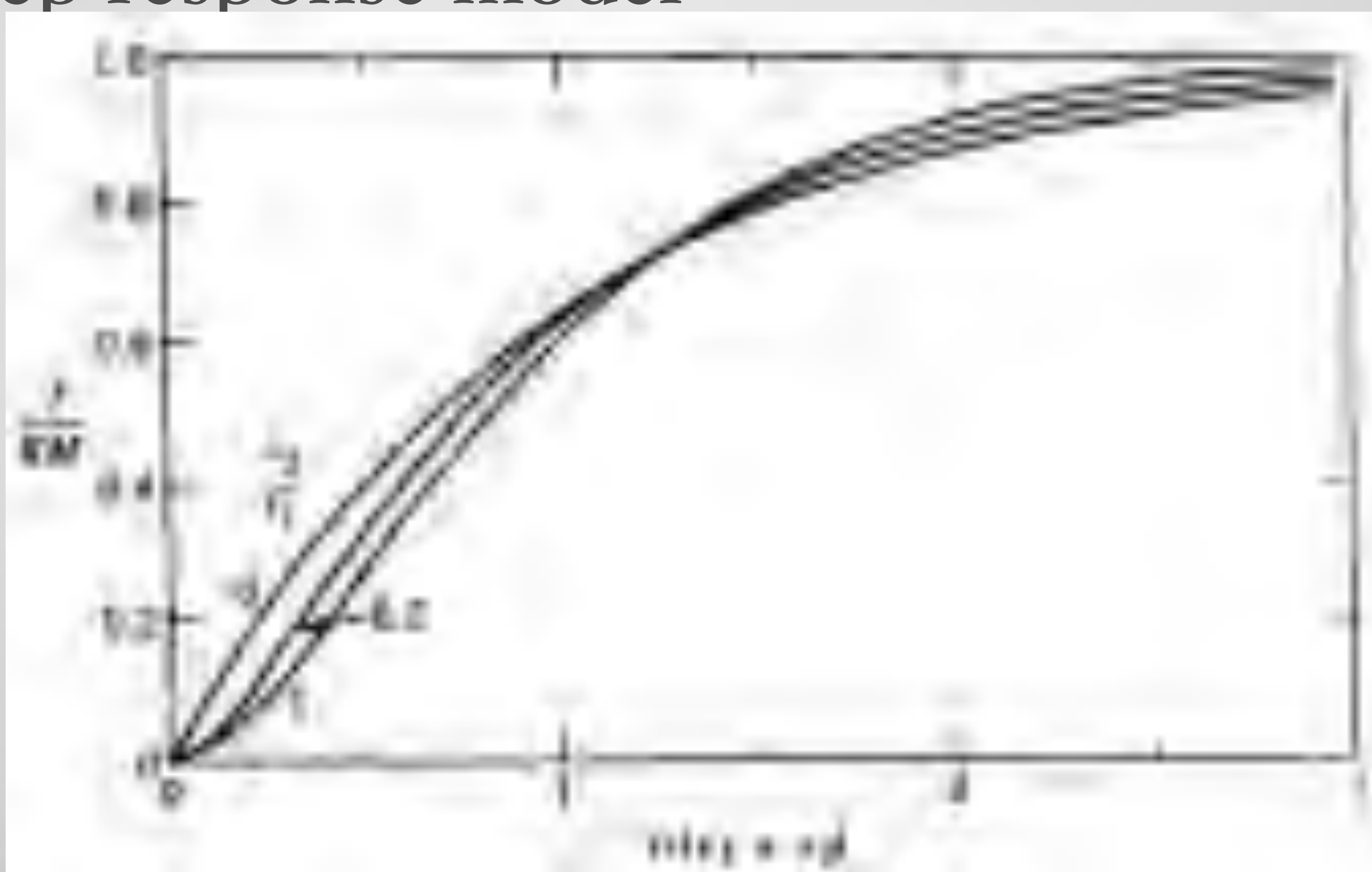
$$G(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$\tau_1 = \frac{\zeta + \sqrt{\zeta^2 - 1}}{\omega_n}, \quad \tau_2 = \frac{\zeta - \sqrt{\zeta^2 - 1}}{\omega_n}$$

- The larger of the two time constants τ_1 is called the dominant time constant.
- Two limiting cases: $\tau_2 / \tau_1 = 0$ where the system becomes first order and
- $\tau_2 / \tau_1 = 1$, the critically damped case.

Over damped and critically damped Models

- Figure below shows the range of shapes that can occur for the step response model



- ***M*** is the total step change

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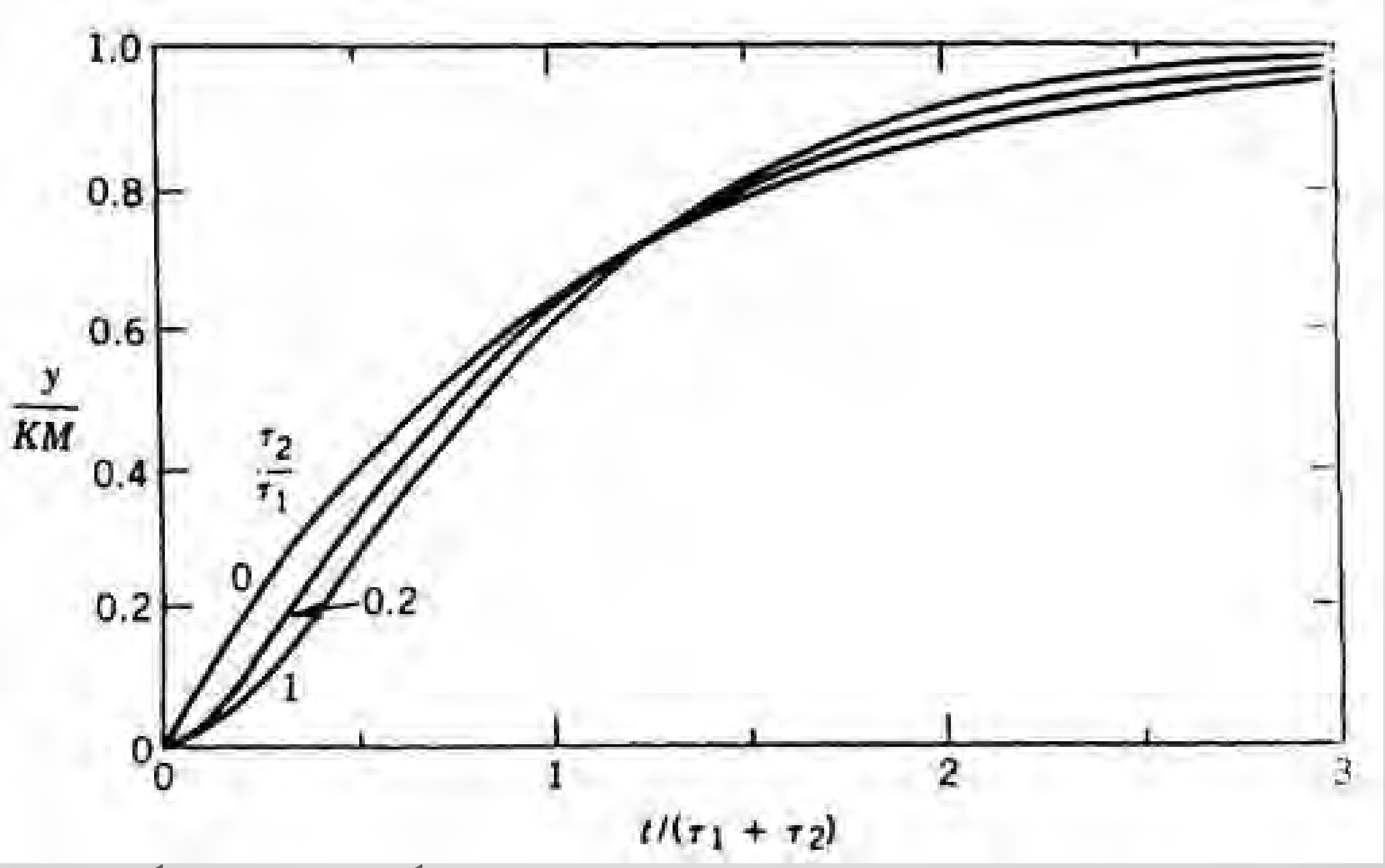
Over damped and critically damped Models

* Figure below shows the range of shapes that can occur for the step response model



* *M* is the total step change

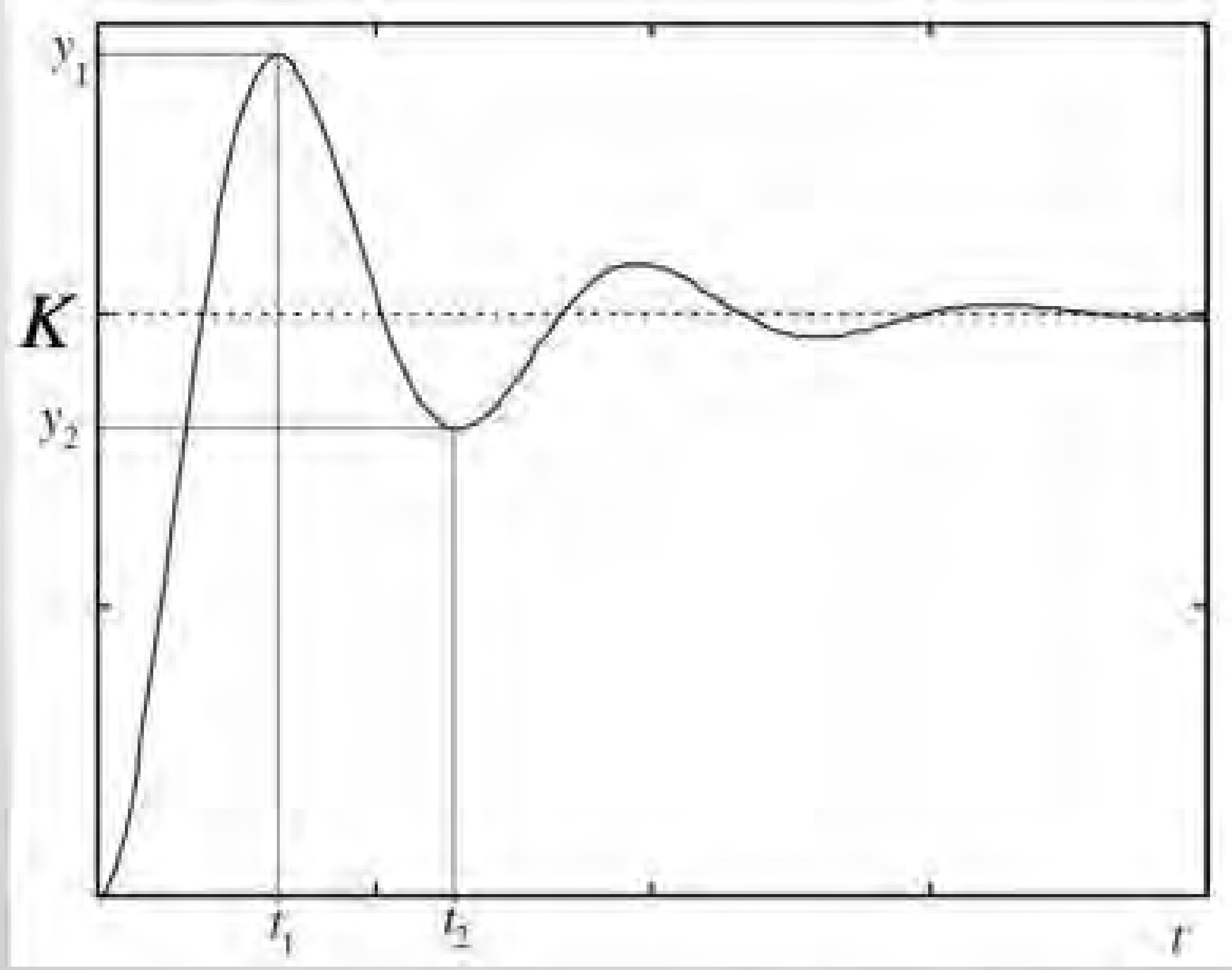
Underdamped Second Order System

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

* The damping $0 \leq \zeta < 1$.
* The ID task is to find K, ω_n and ζ.
* The process static gain is as in the previous case given as the new steady-state value of the process output $K = y(\infty)$.

Underdamped Second Order System

* Given are points [t1, y1], [t2, y2] and the steady state output y(∞).
* we will use the fact that the derivative of the step response with respect to times is in the points ***tn*** (local extrema) zero.



Underdamped Second Order System

* The step response is of the form

$$y(t) = K\left(1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t\right)\right)$$

Prove

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

* The derivative of *y(t)* with respect to time is given as

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t$$

Prove

* The slope in above is zero at local extrema:

$$\dot{y}(t_n) = 0 \rightarrow \sin\omega_d t_n = 0$$

$$t_n = \frac{n\pi}{\omega_d} \quad , \quad n = 0, \pm 1, \pm 2, \ldots \qquad (1)$$

Underdamped Second Order System

- Substituting t_n in *y(t)*

$$y(t_n) = K\left(1 - \left(e^{-\zeta\omega n t_n} \cos\omega_d t_n + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t_n\right)\right)$$

- At n=1 in eq.(1) (Overshoot):

$$y(t_1) = K(1 + e^{\frac{-\zeta\omega_n\pi}{\omega_d}})$$

$$y(t_1) = K(1 + M_p), \qquad M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

- At n=2:

$$y(t_2) = K(1 - e^{-2\frac{\zeta\omega_n\pi}{\omega_d}})$$

$$y(t_2) = K(1 - M_p{}^2)$$

Underdamped Second Order System

* The identification procedure is then as follows:

1. $K = y(\infty),$

2. $y_1 = K(1 + M_p), y_2 = K(1 - M_p{}^2) \Rightarrow M_p = \dfrac{y_1 - y_2}{y_1}$

3. $M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \Rightarrow \quad \zeta = \left| \dfrac{\ln M_p}{\sqrt{\pi^2 + (\ln M_p)^2}} \right|$

Prove

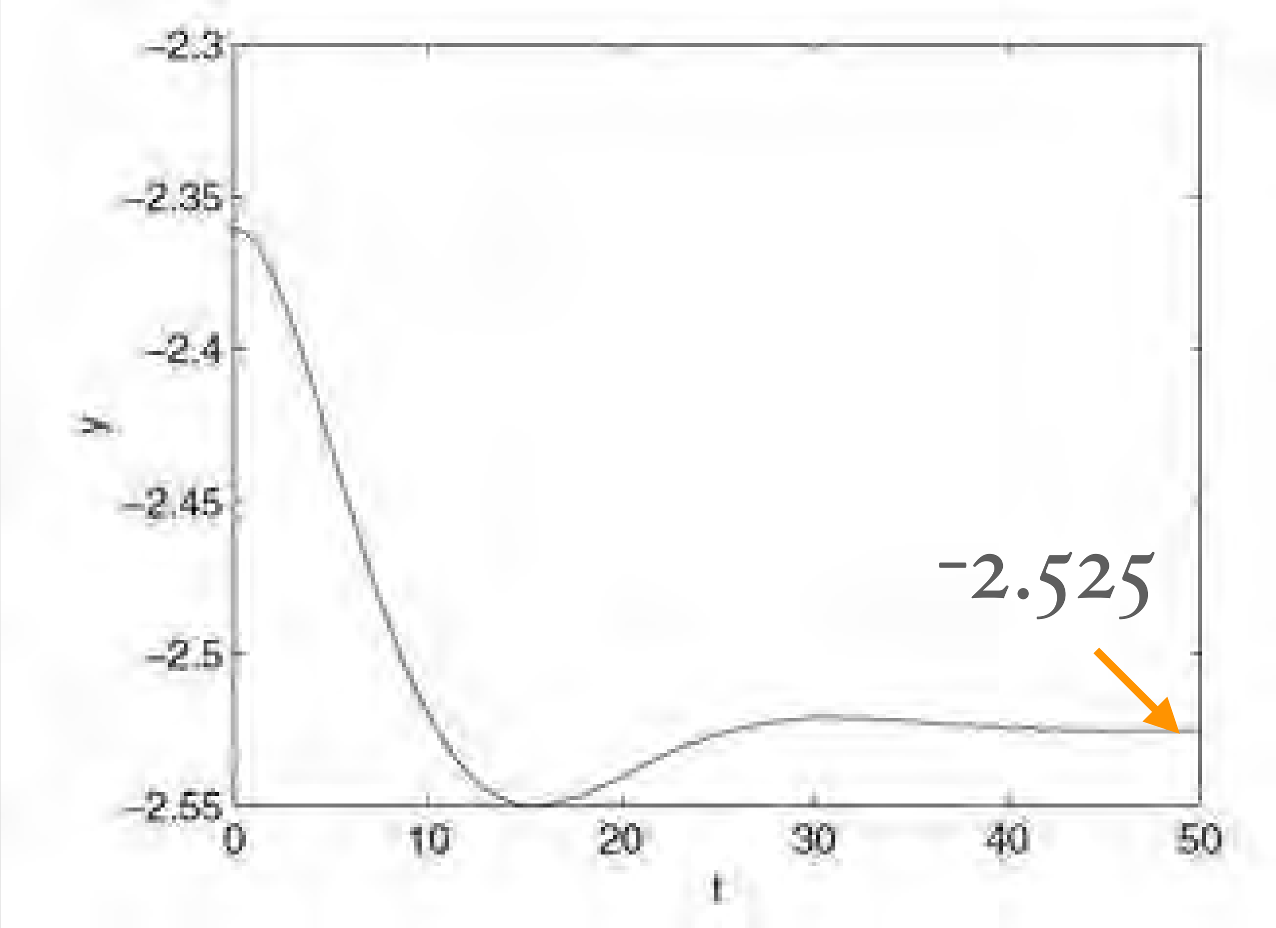
4. $t_1 = \dfrac{\pi}{\omega_d}, t_2 = \dfrac{2\pi}{\omega_d} \qquad \omega_n = \dfrac{\pi}{(t_2 - t_1)\sqrt{1 - \zeta^2}}, \tau = \dfrac{1}{\omega_n}$

Example: Underdamped System

Example: SI of Second-order System

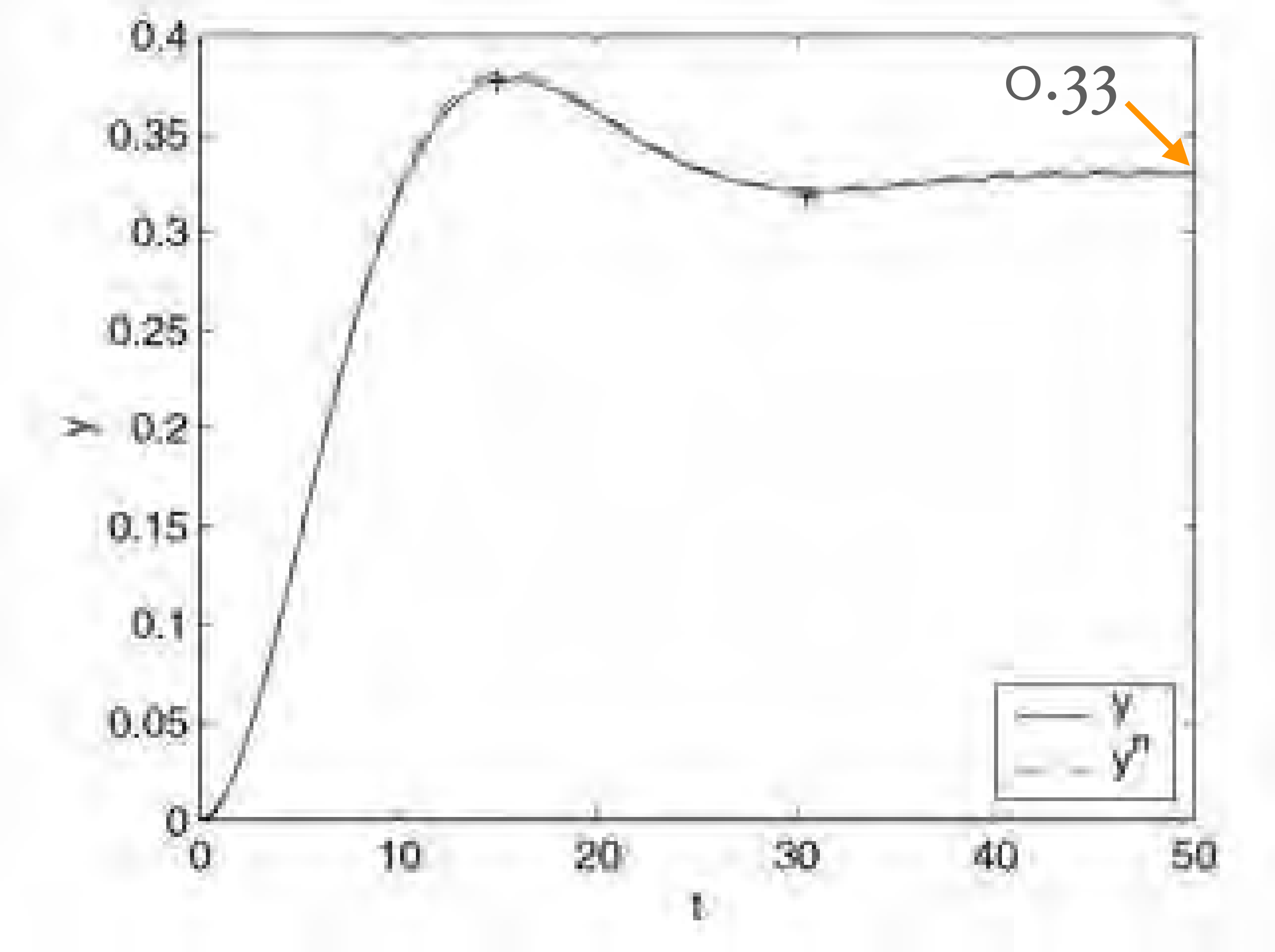
* Consider a measured step response shown that has been measured from the steady-state characterized by the input variable at the value u(0) = 0.2 changed to the value u(∞) = -0.3.

* Such a step response can be obtained for example from a U-tube manometer by a step change of the measured pressure.



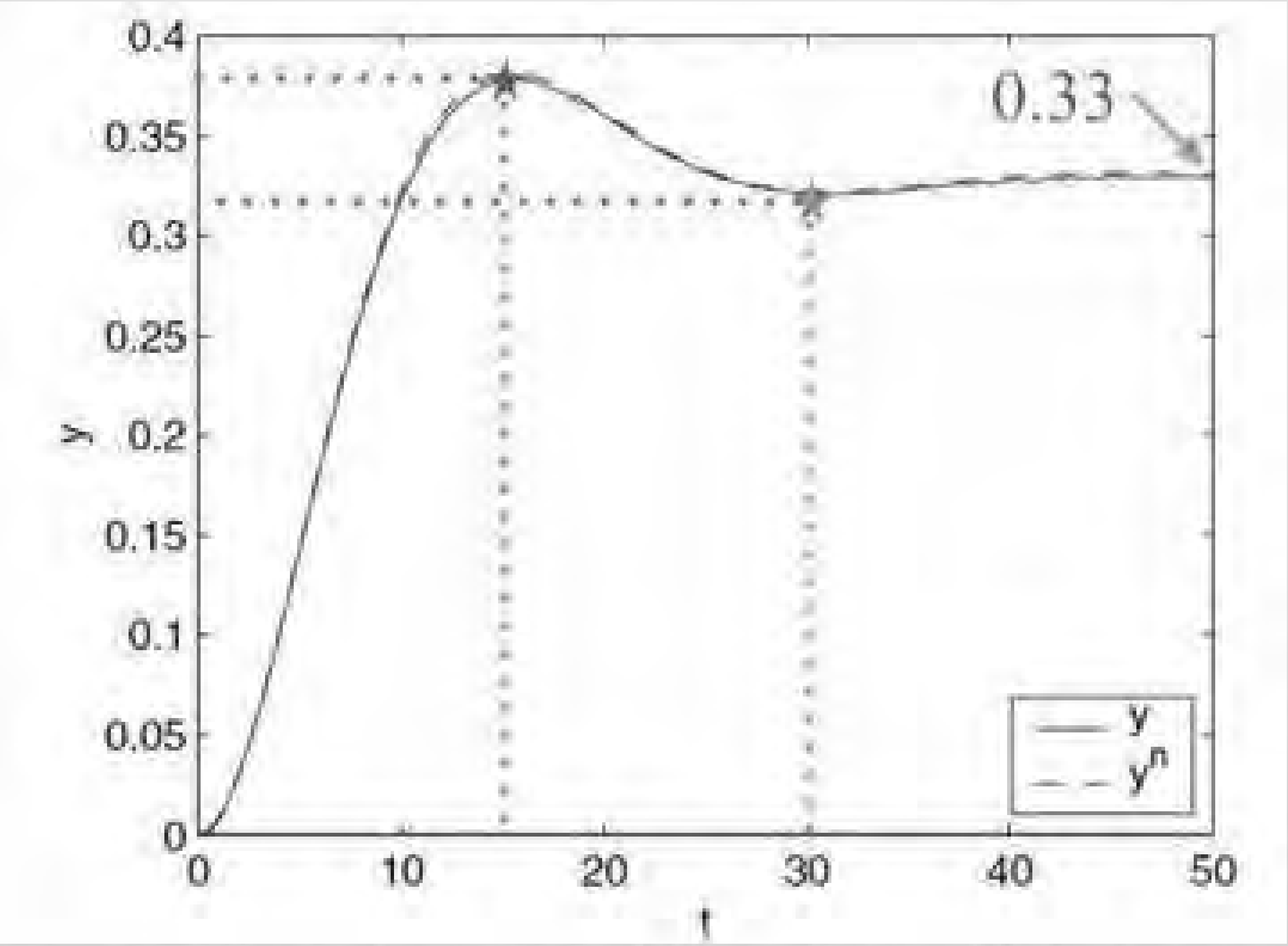
Example: SI of Second-order System

* The measured step response is first shifted to the origin by a value of y0 = −2.3608 and then normalised − divided by the step change of theinput Δu = 0.5.



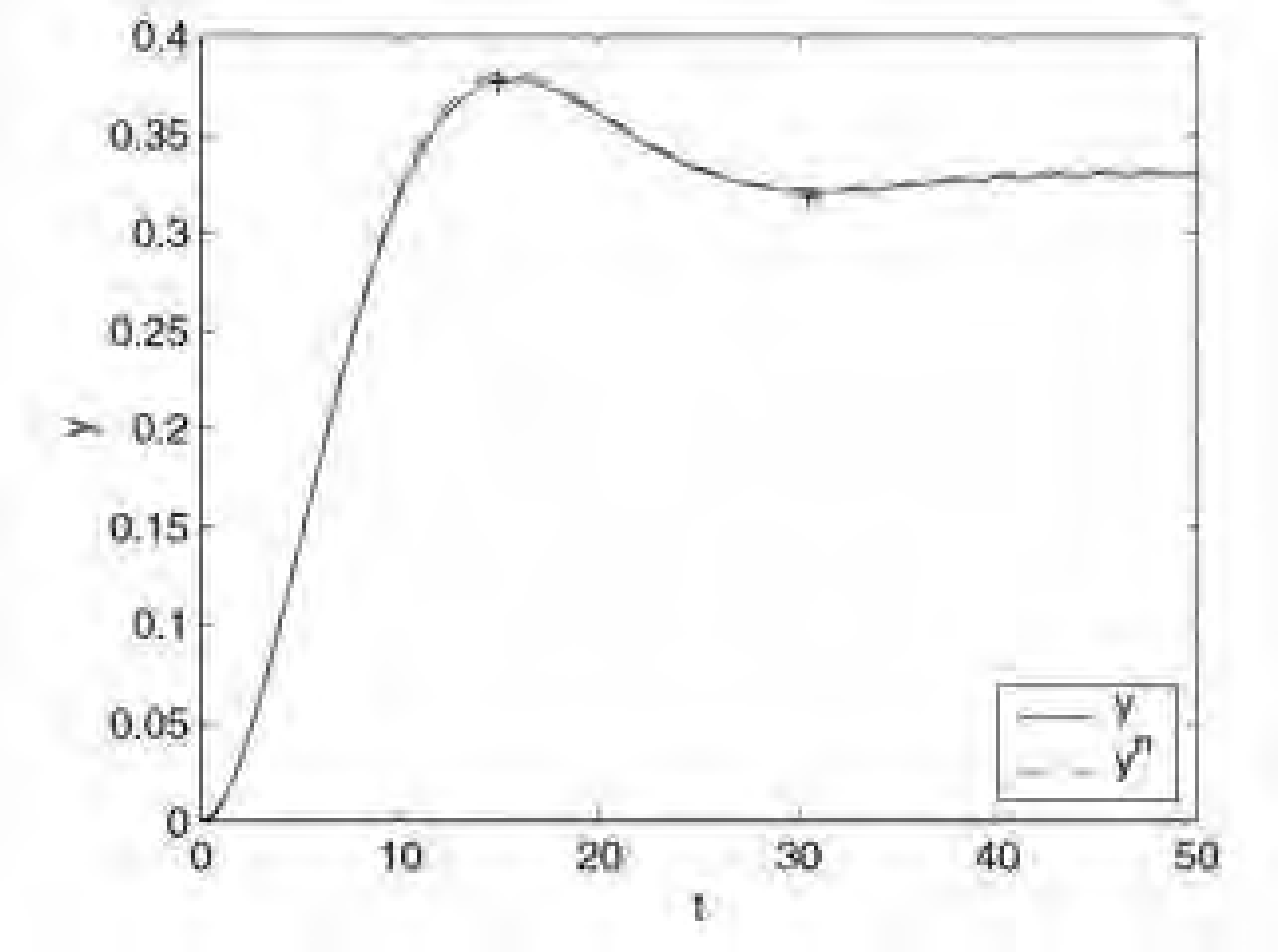
* The measured step response is first shifted to the origin by a value of y0 = −2.3608 and then normalised − divided by the step change of theinput Δu = 0.5.

* The values of the first maximum and minimum are found as [15.00; 0.38] and [30.50; 0.32], respectively



* K = 0.33, ζ = 0.51, and τ = 4.22.

$$G(s) = \frac{0.33}{17.8084s^2 + 4.3044s + 1}$$



Fitting Second -order Models

Smith's Method

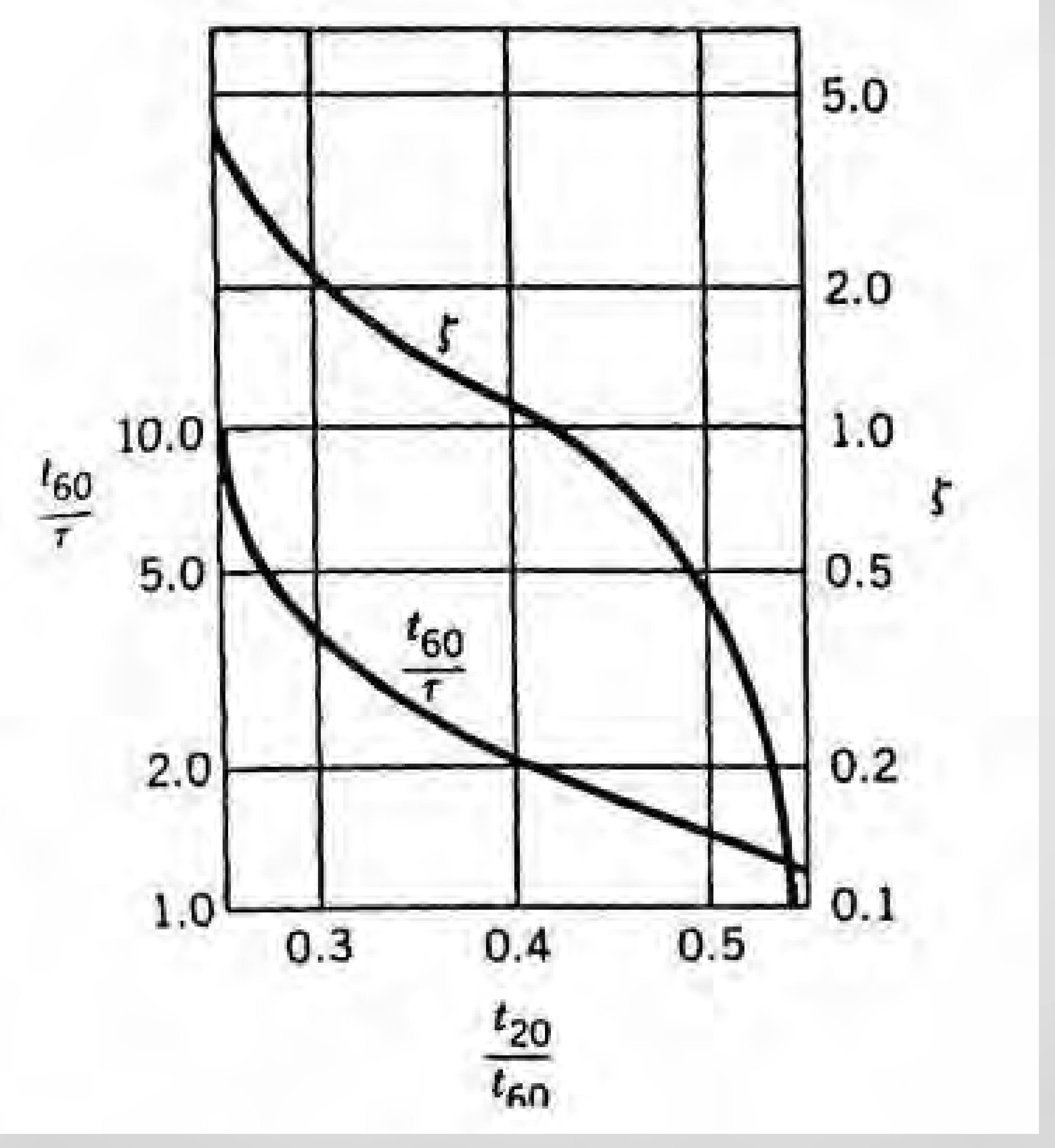
Smith's Method

* The assumed model:

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

1. Determine t_{20} and t_{60} from the step response.
2. Find ζ and $\frac{t_{60}}{\tau}$ From the following Figure.
3. Find τ

Smith's method



relationship of ζ and τ to t_{20} and t_{60}.

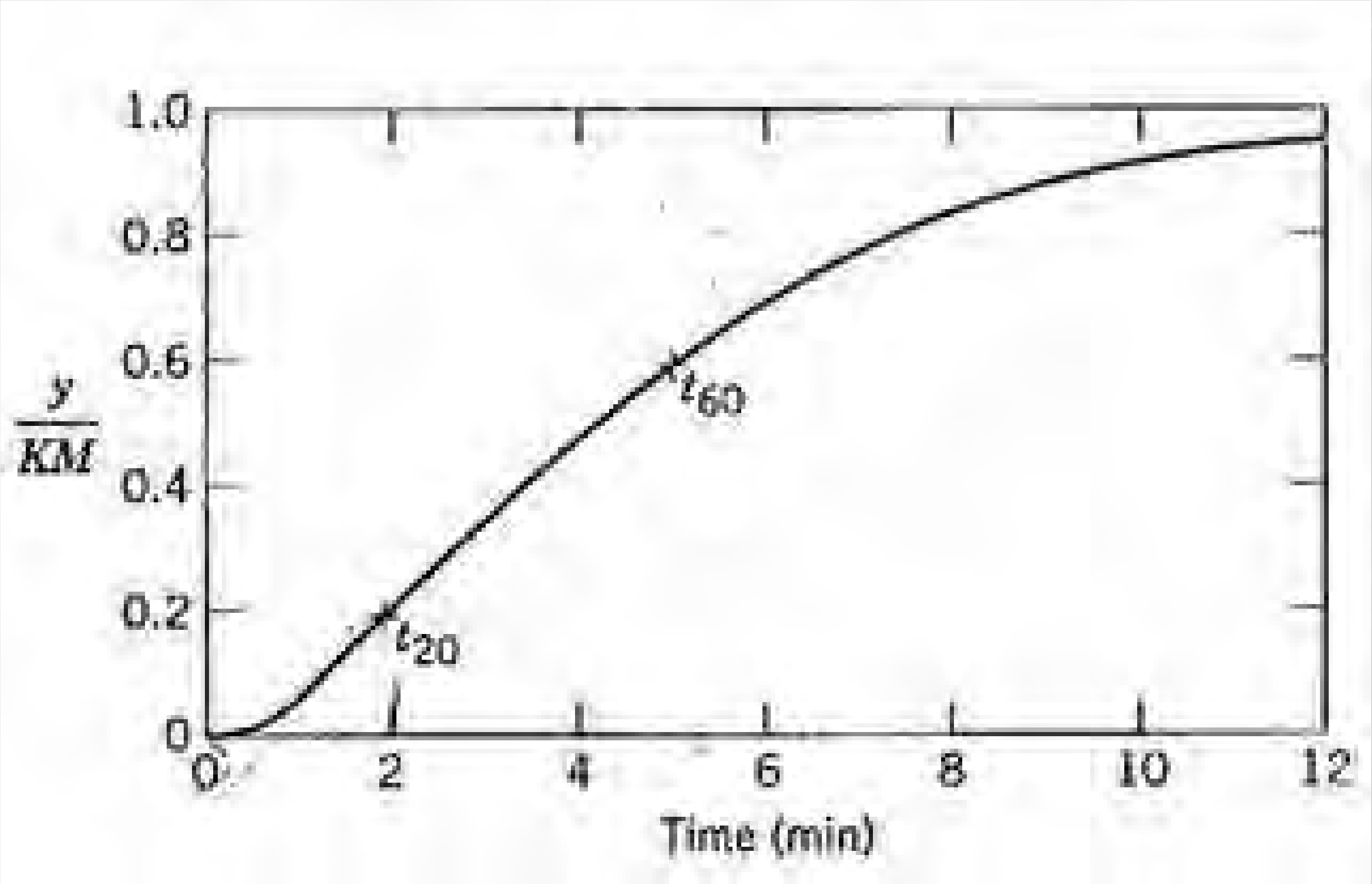
Example

t_{20}=1.85min.; t_{60}=5min.

$$\frac{t_{20}}{t_{60}} = 0.37 \Rightarrow \zeta = 1.3,$$

$$\frac{t_{60}}{\tau} = 2.8 \Rightarrow \tau = 1.79\,\text{min}.$$

$$\tau_1 = 3.81\,\text{min}.; \tau_2 = 0.84\,\text{min}.$$



because the system is overdamped, the two time const. can be calculated (see slide 56).

$$G(s) = \frac{1}{3.2s^2 + 3.58s + 1} = \frac{1}{(3.81s + 1)(0.84s + 1)}$$

Second-Order Models &Time DELAY

* When fitting 2nd order models, the time delay must be estimated with caution.
* When $\tau_2 / \tau_1 = 1$, there is an inflection point, tangent method indicates a time delay.
* Visual determination of time delay is recommended by graph. estimation and trial and error to have a good fit.
* Hence, the model

$$G(s) = \frac{Ke^{-\theta s}}{\tau^2 s^2 + 2\zeta\tau s + 1} \qquad \theta = T_d$$

Identification from Impulse Responses

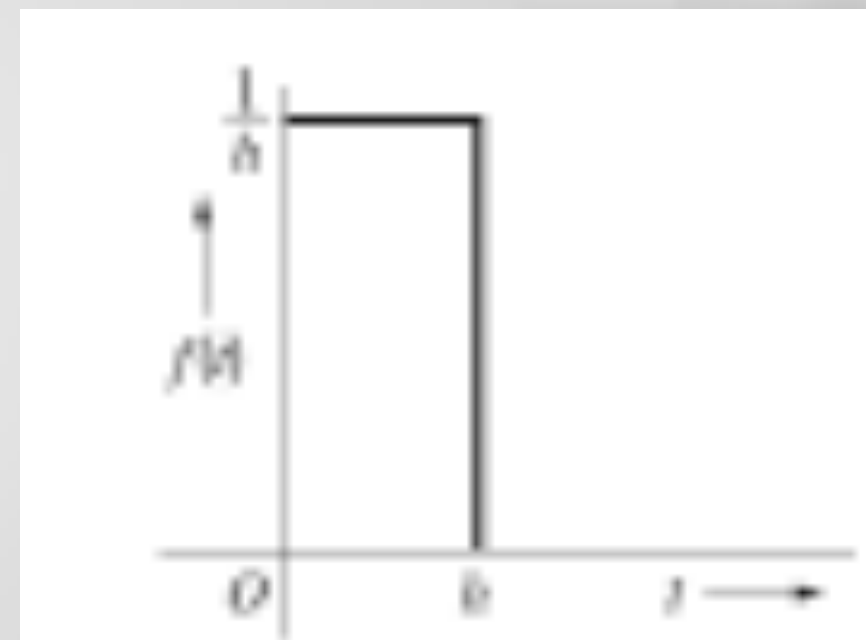
A classical Method

Impulse response identification

- A rectangular function can be used to depict the opening and closing of a valve regulating flow into a tank.

- A unit rectangular pulse can expressed as:

$$f(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{b} & 0 < t < b \\ 0 & t > b \end{cases}$$



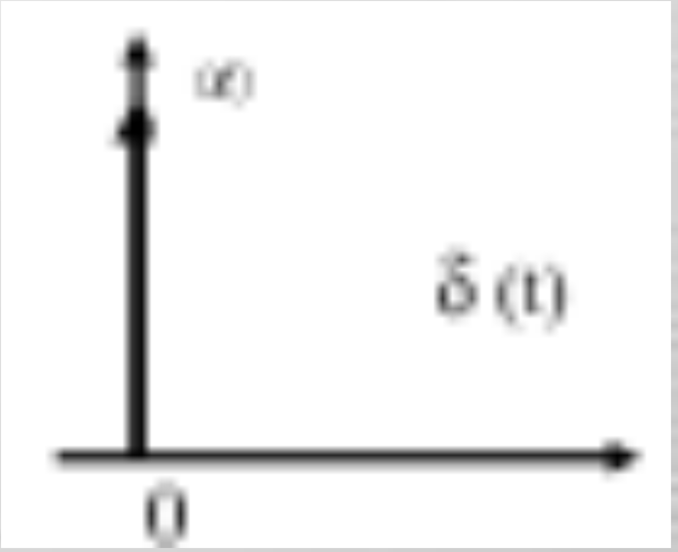
- It is clear that f (t) may be represented by the difference of two functions

$$f(t) = \frac{1}{b}\left[u(t) - u(t-b)\right]$$

Impulse response identification

- If we allow h to shrink to zero, we obtain a new function which is zero everywhere except at the origin, where it is infinite.
- However, it is important to note that the area under this function always remains equal to unity.
- Theoretically, the impulse function, which is denoted with δ (t), can be defined as:

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} (\infty) & t = 0 \\ 0 & t > 0 \end{cases}$$



Impulse response identification

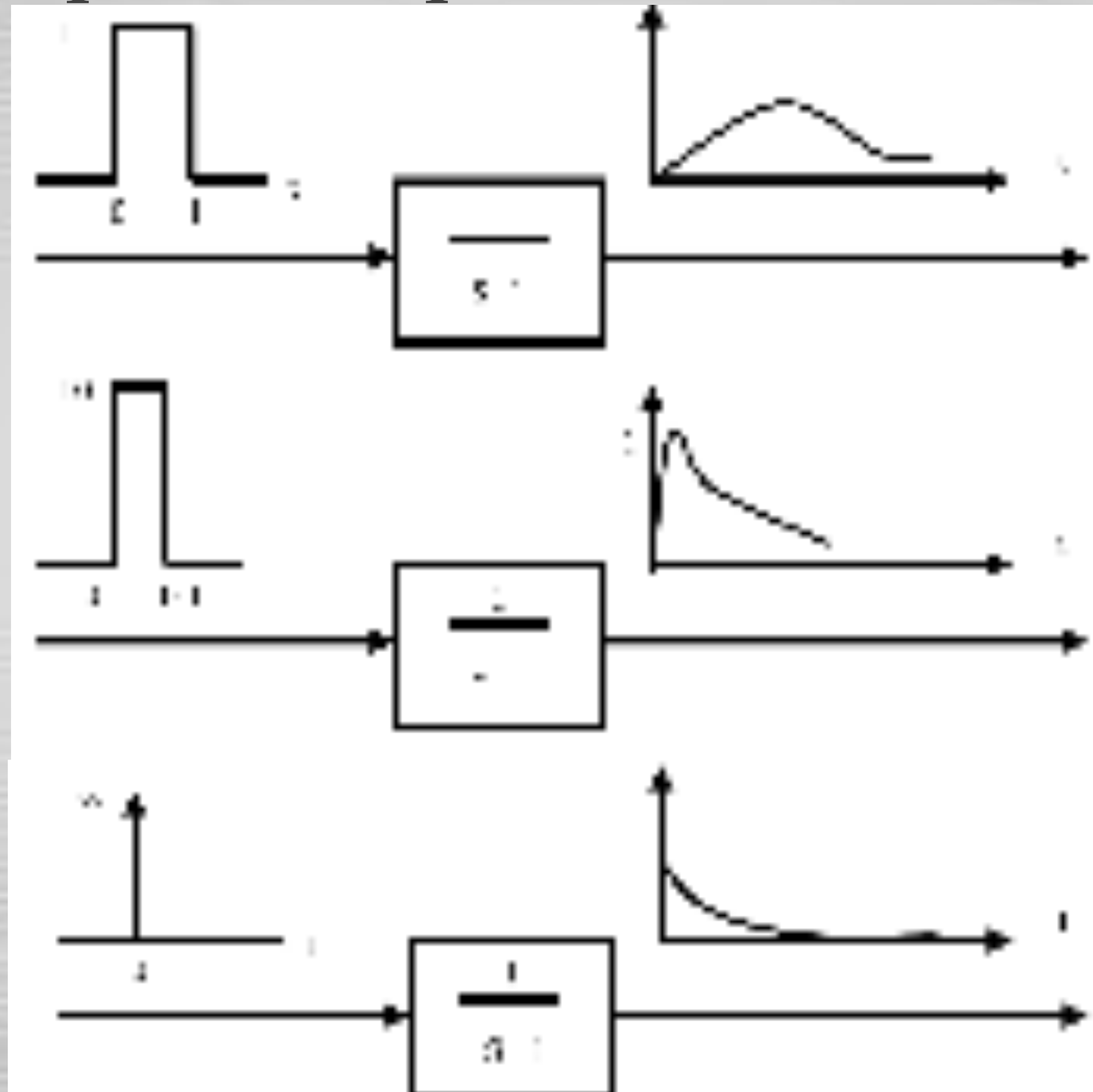
- A direct approach is to apply an impulse input and observe the response.

- Consider the response of a first order system to a pulse input of amplitude (1 / h) and a duration h.

- If the time duration of the input is sufficiently small compared with the system time constant T, then the response is approximately a unit impulse response.

- Note that if $h < 0.1T$, the response of the system is almost identical to the unit impulse response.

Impulse response identification



Impulse response identification

- For a system TF G(s)

$$Y(s) = G(s)X(s), \;\; impulse \;\; I/P: \qquad R(s) = 1$$

$$Y(s) = G(s)$$

Impulse Response

First-order System

Impulse response of a First Order System

- For a First order system:

$$G(s) = \frac{K}{Ts+1} = \frac{\frac{K}{T}}{S+\frac{1}{T}}$$

$$y(t) = g(t) = \frac{K}{T} e^{-\frac{t}{T}}$$

Impulse response of a First Order System

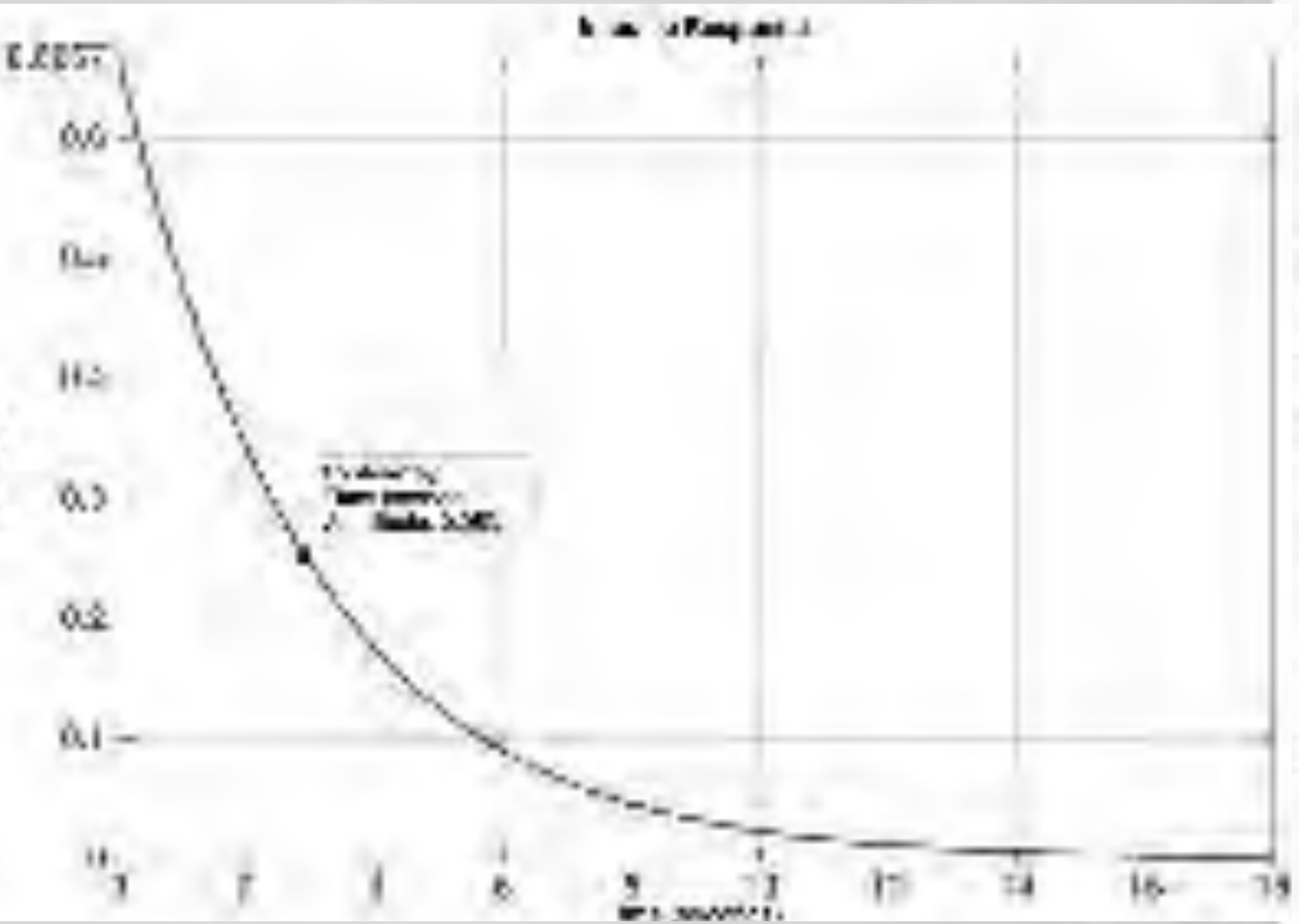
- Find K from the initial value of the variable y (t):

$$y(0) = \frac{K}{T}$$

- Find T by setting t=T in y(t):

- $$y(T) = \frac{K}{T} e^{-1} = 0.368 \frac{K}{T}$$

Example: Find the transfer function of the following impulse response:



$$y(0) = \frac{K}{T} = 0.6667$$

$$y(T) = 0.368 * \frac{K}{T}$$

$$y(T) = 2.453$$

$$\Rightarrow T = 2.9 \sec.$$

$$\frac{K}{T} = 0.6667$$

$$\Rightarrow K = 1.933$$

$$G(s) = \frac{1.933}{3s+1}$$

Impulse Response

Second-order System

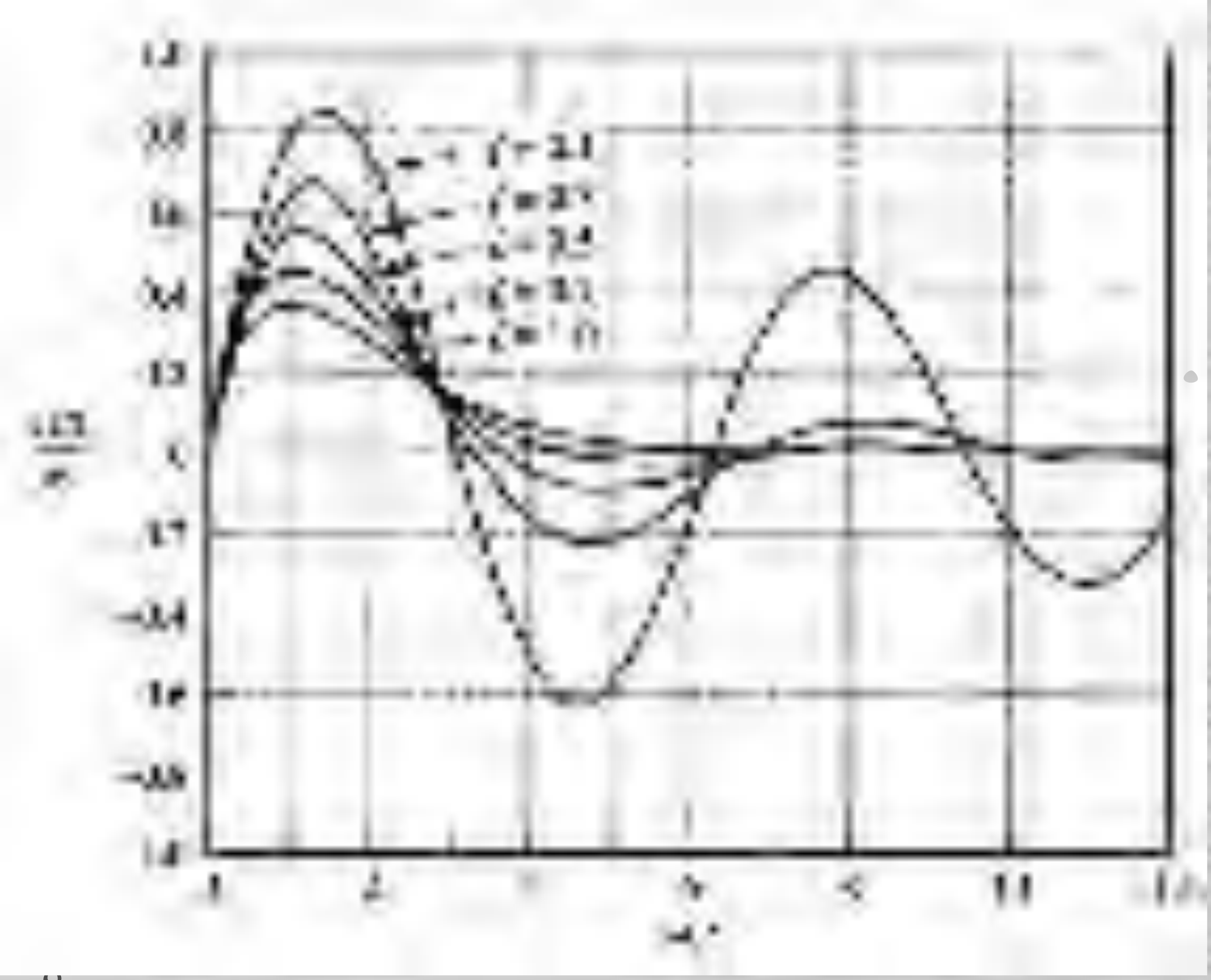
Impulse response of a 2nd order System

- The unit impulse response of an underdamped 2nd order system is

$$Y(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}, \qquad 0 < \zeta < 1$$

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t),$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$



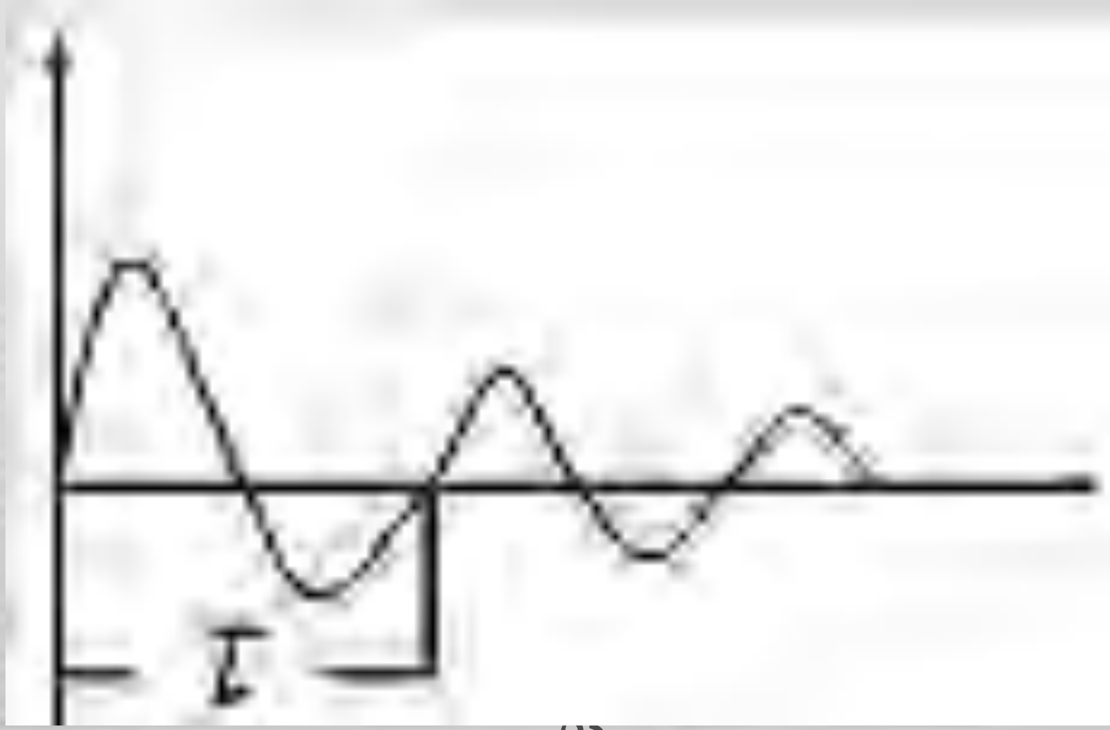
Impulse response of a 2nd order System

- The parameters are found as follows:

$$\omega_d = \frac{2\pi}{\tau},$$

- τ is the period of <u>one oscillation</u>, then find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$$



Impulse response of a 2nd order System

- The damping ratio can be found by log-decrement method

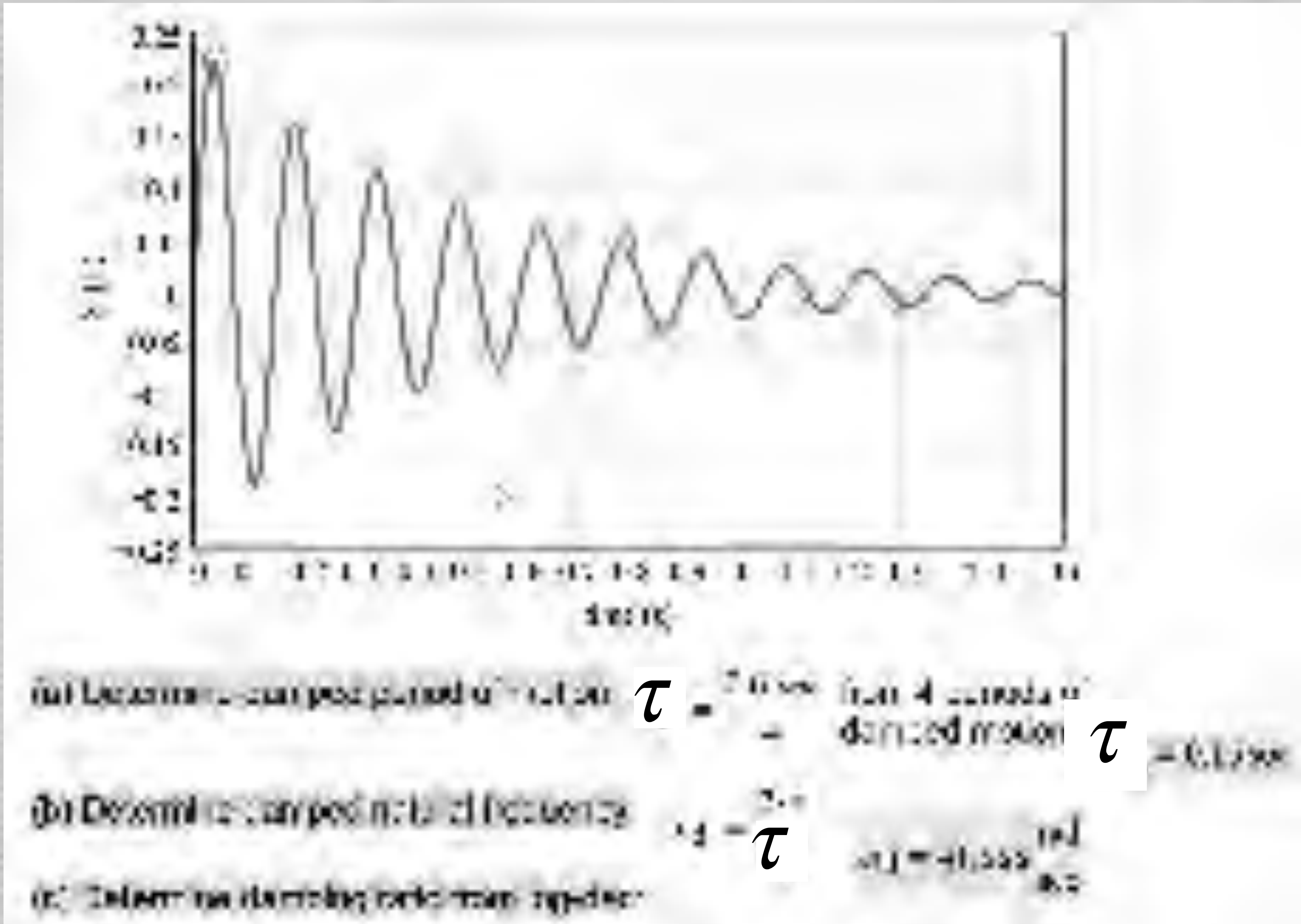
$$\zeta = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\delta}\right)^2}}$$

$$\delta = \frac{1}{n} \ln \frac{x(t)}{x(t + nT)}$$

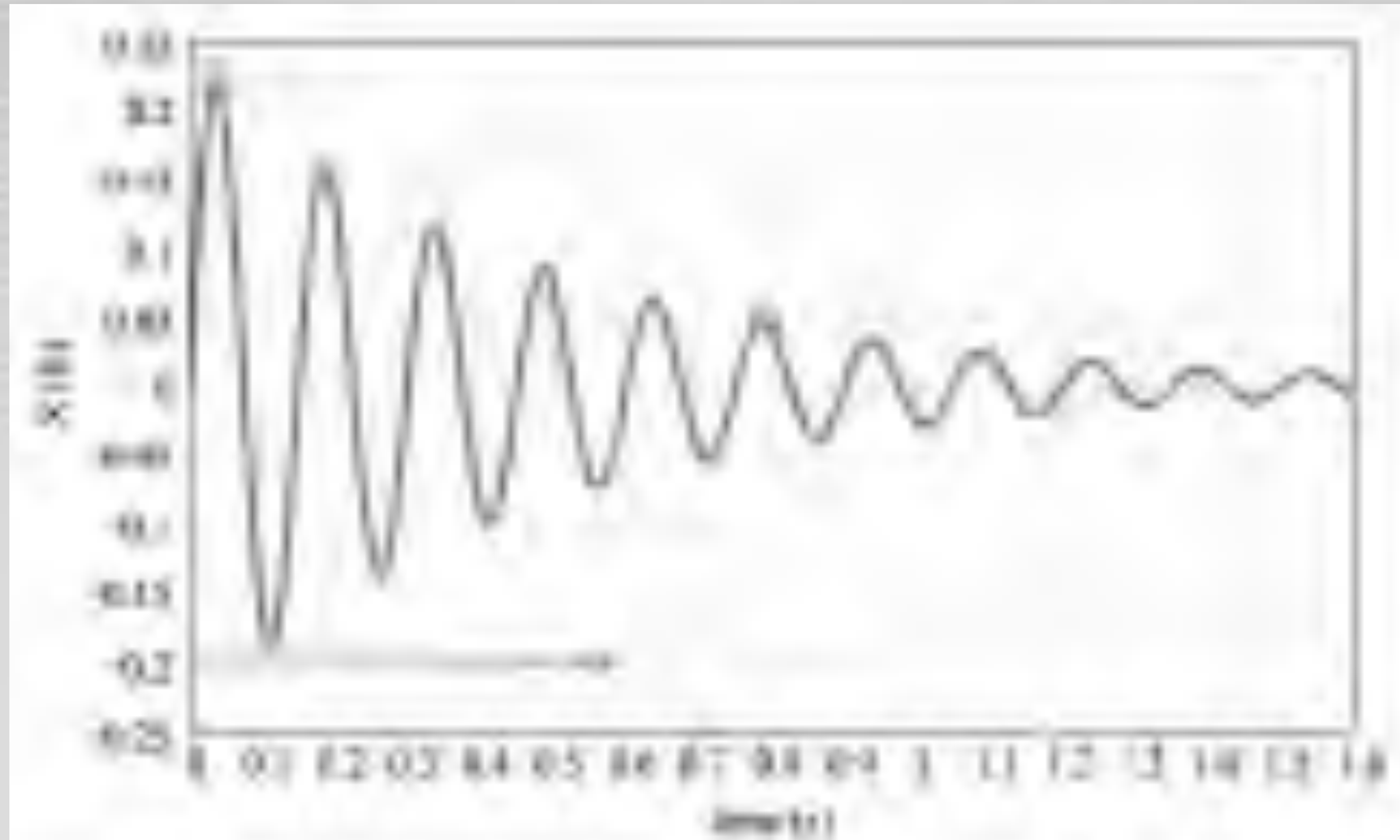
- where *x(t)* is the <u>amplitude at time *t* and *x(t+nT)*</u> is the amplitude of the peak n periods away, where n is any integer number of successive, positive peaks.

Example: Underdamped System

Example



Example



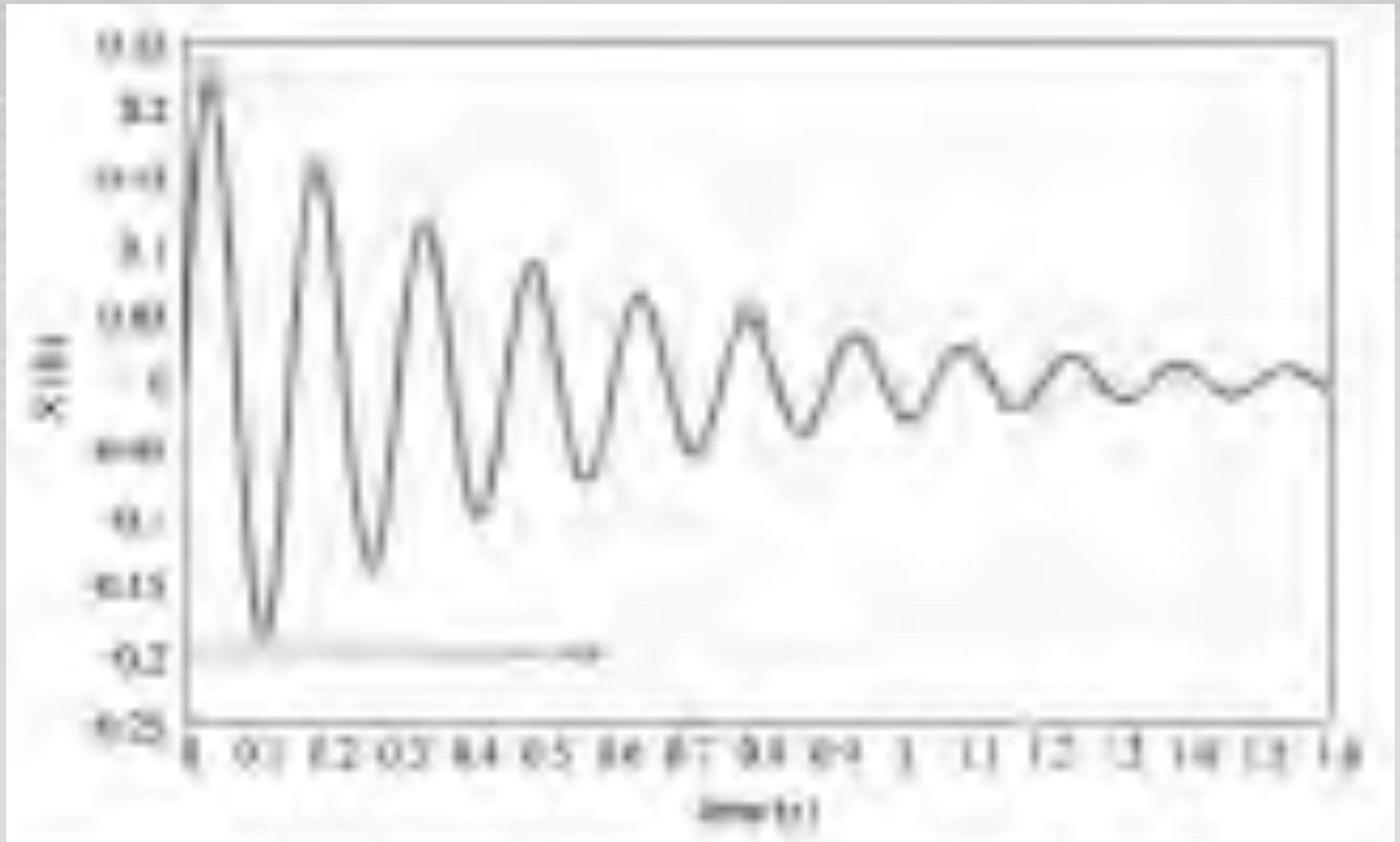
Select two amplitudes of motion (well spaced) and count number of periods in between

$X_0 := 0.23 \cdot ft$ after $n := 5$ periods $X_n := 0.05 \cdot ft$

Log-dec is derived from ratio: $\delta := \frac{1}{n} \cdot \ln\left(\frac{X_0}{X_n}\right)$ $\delta = 0.305$

from log-dec formula $\delta = \frac{2 \cdot \pi \cdot \zeta}{\left(1 - \zeta^2\right)^{0.5}}$ $\zeta := \frac{\delta}{\left(4\pi^2 + \delta^2\right)^{.5}}$ $\zeta = 0.049$

Example



Impulse response of a 2nd order System

critically damped case ($\zeta = 1$):

$$y(t) = \omega_n^2 t e^{-\omega_n t} \qquad t \geq 0$$

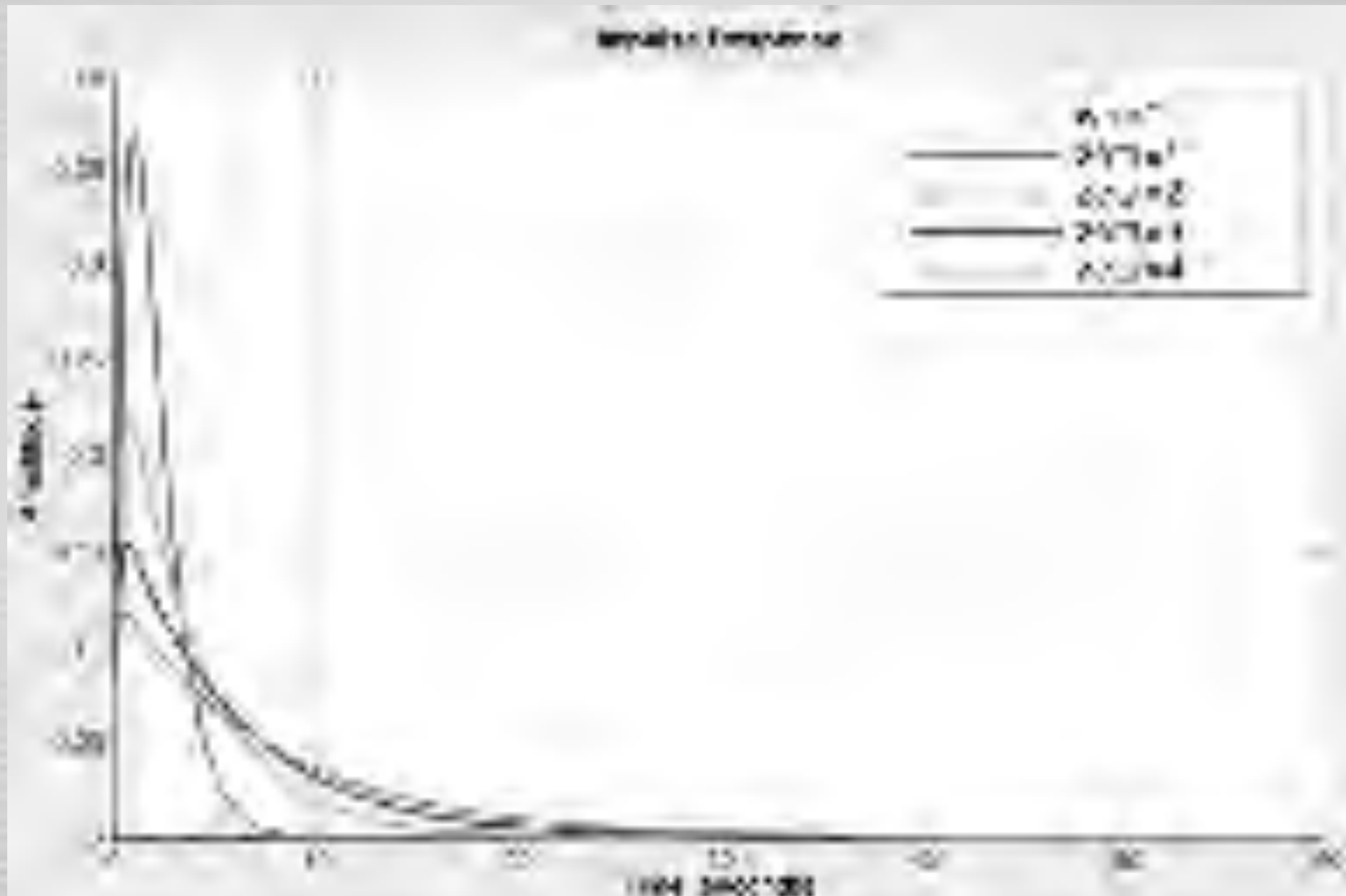
overdamped case ($\zeta > 1$):

$$y(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_2 t} \qquad t \geq 0$$

where

$$s_1 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$
$$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

Impulse response of a 2nd order System



System Identification

A Third-year Course for Control and Mechatronics Engineering

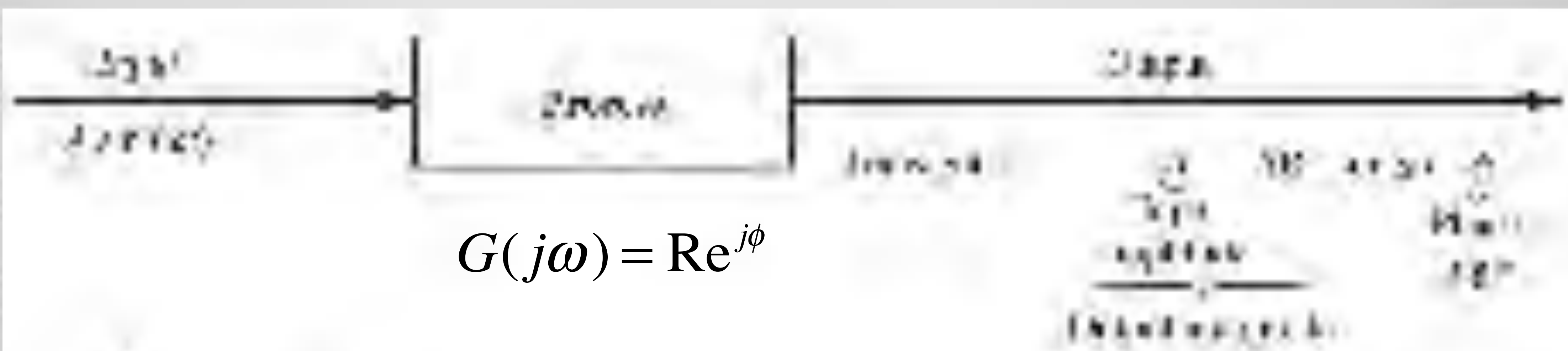
By Dr. Taghreed M. MohammadRidha

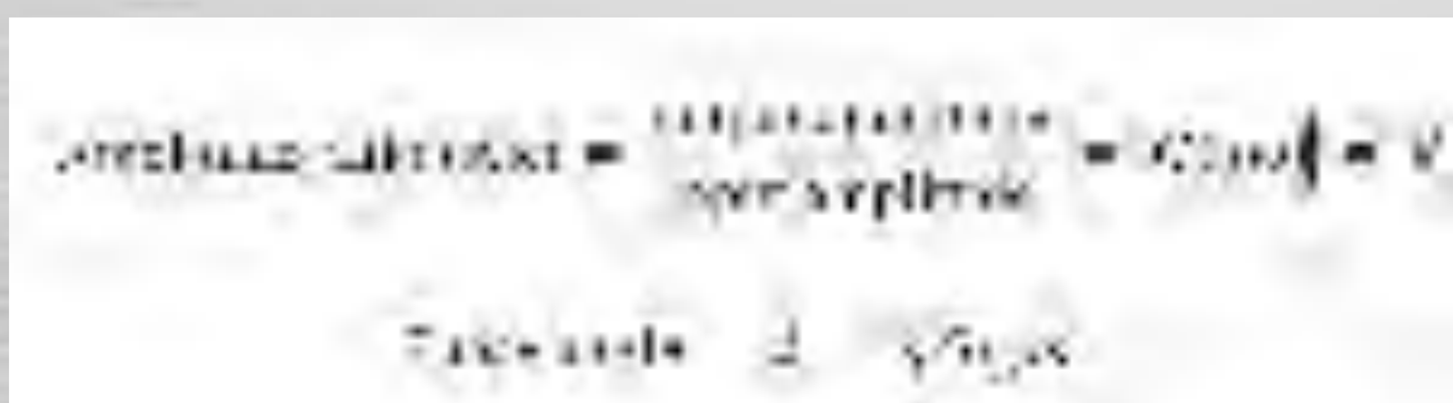
Identification From Frequency Response

A classical Method

Frequency Response

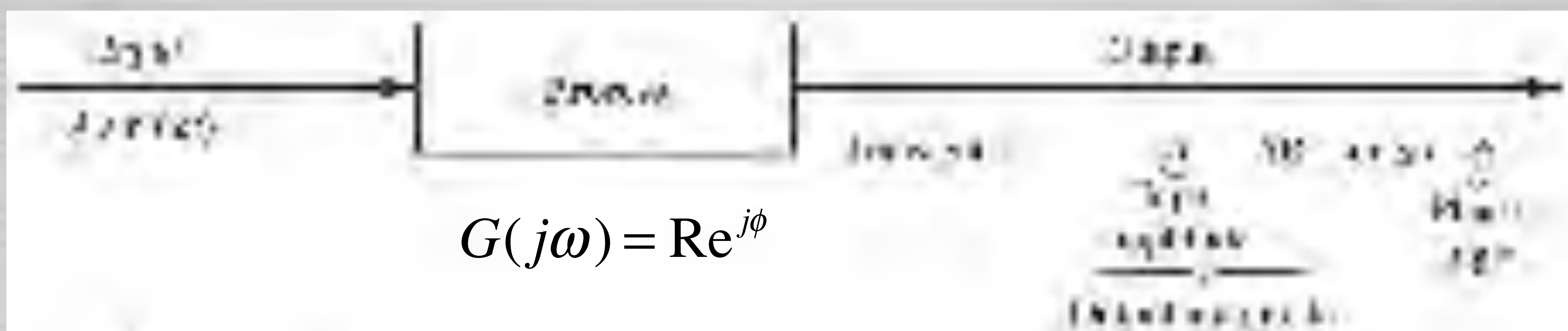
- In sinusoidal circuit analysis, if we left the amplitude of the sinusoidal source remain constant and vary the frequency, we obtain the circuit's <u>frequency response.</u>

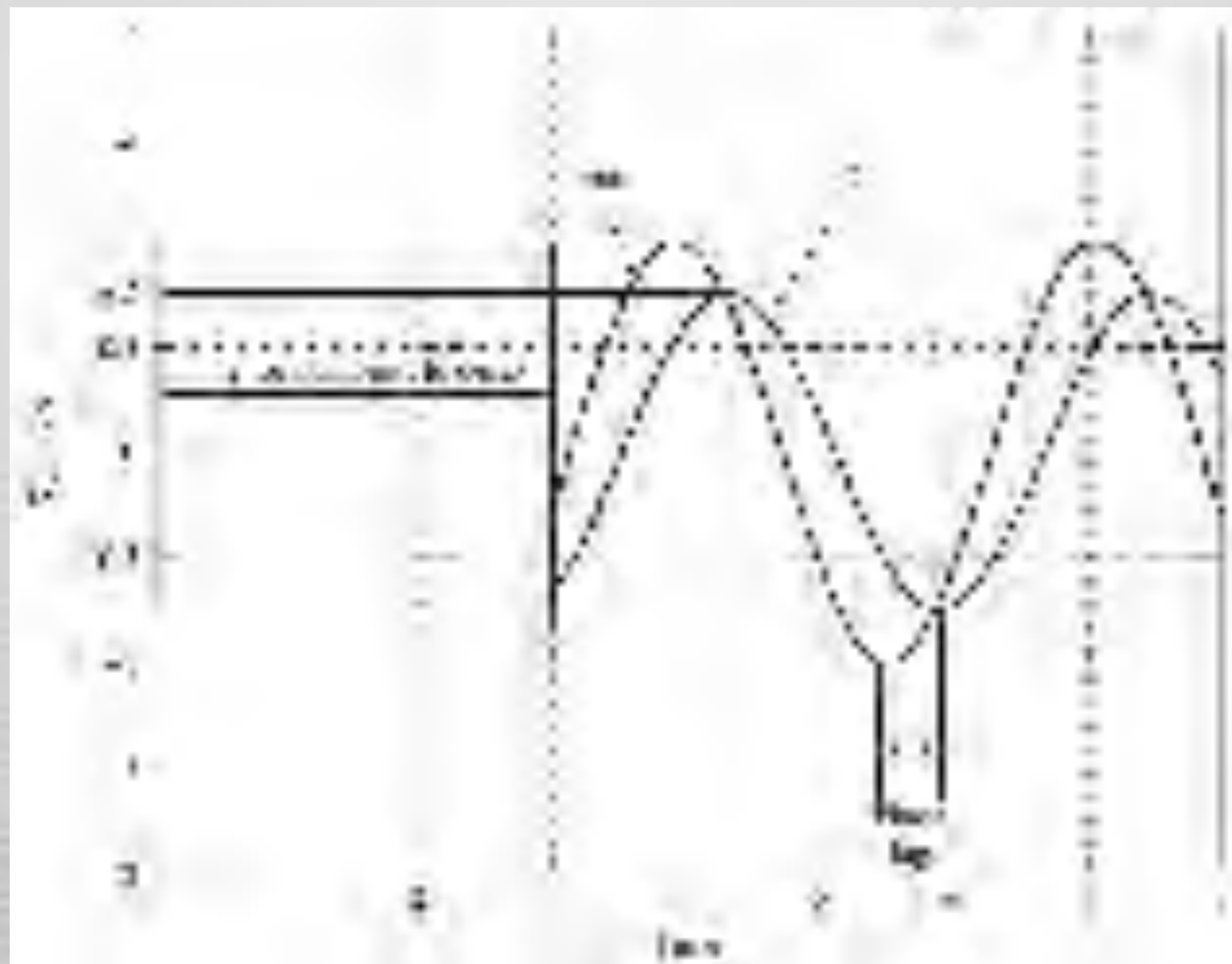




$$G(j\omega) = \mathrm{R}e^{j\phi}$$

Frequency Response [5]

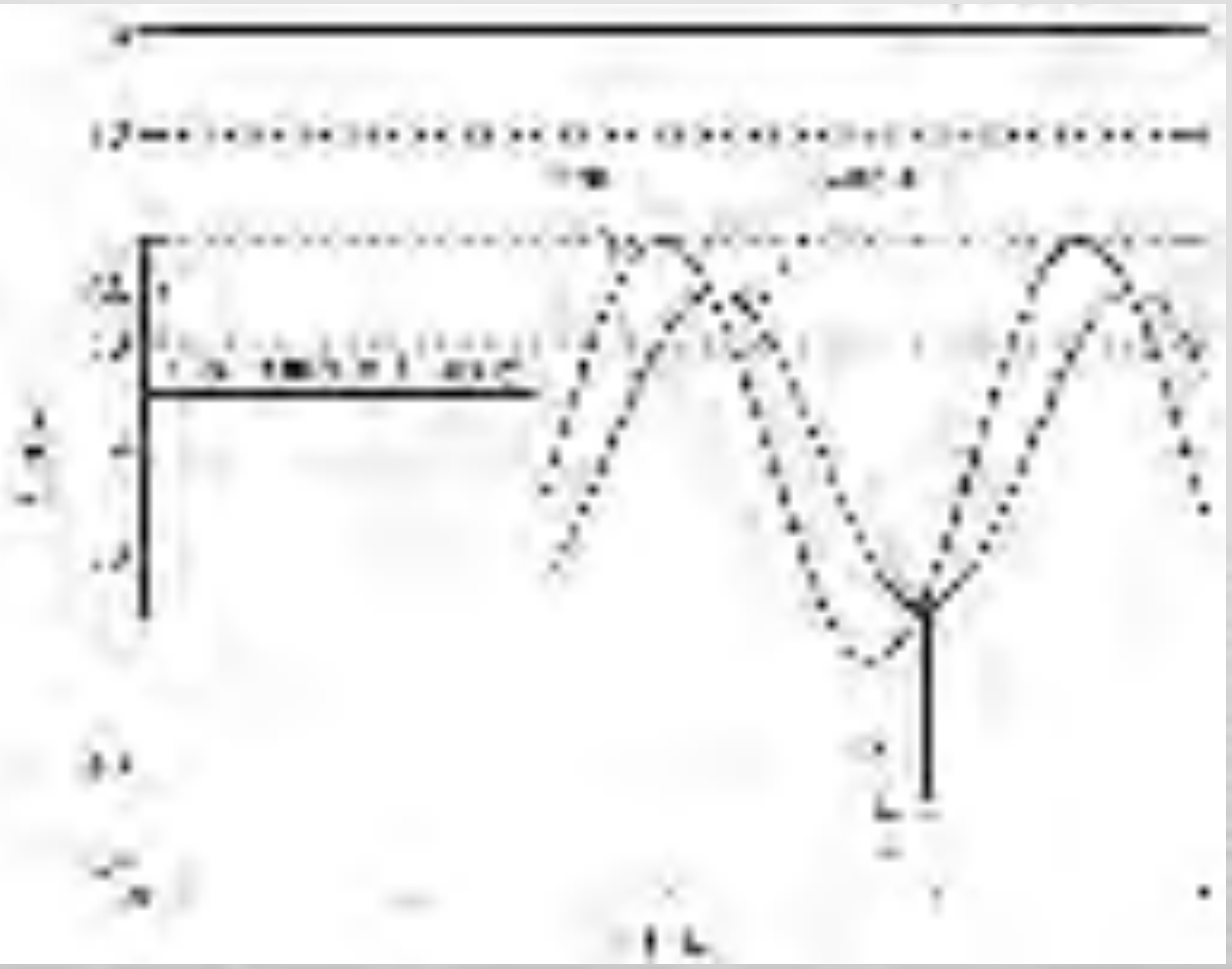




Frequency Response

- The frequency response may be regarded as a complete description of the sinusoidal steady-state behavior of a circuit as a function of frequency.

- <u>The frequency response</u> of a circuit is the variation in its behavior with change in signal frequency.



Identification From Frequency Response

Bode Plot

Bode Plot

- Bode plots are semi log plots of the magnitude (in *decibels*) and phase (in *degrees*) of a transfer function versus frequency.

- The frequency range required in frequency response is often so wide that is inconvenient to use a linear scale for the frequency axis.

- Here, there is a more systematic way of locating the important features of the magnitude and the phase plots of transfer function.

- On a log scale (e.g., dB), the product turns into a sum. Thus, if we plot the behavior of each term, we can then simply *add* the plots to find the total behavior.

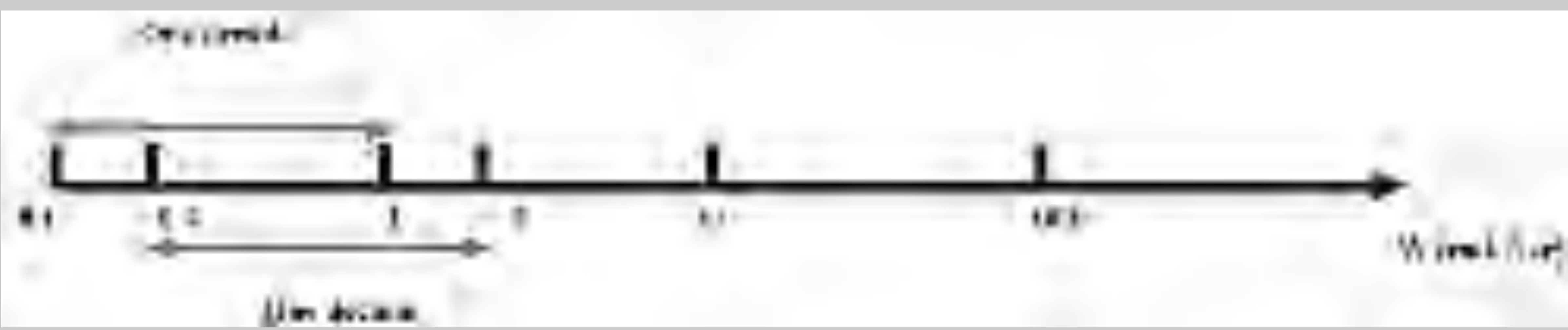
- For these reasons, it has become standard practice to use a logarithmic scale for the frequency axis and a linear scale in each of the separate plots of magnitude and phase.

Identification by Bode Plot

- A Bode plot is a standard format for plotting frequency response of LTI systems.
- Sine wave inputs are applied to the systems.
- Steady state output is observed (Magnitude ratio R and phase Φ).
- Use R and Φ plots to estimate the various break frequencies (poles and zeros) of the transfer function.
-

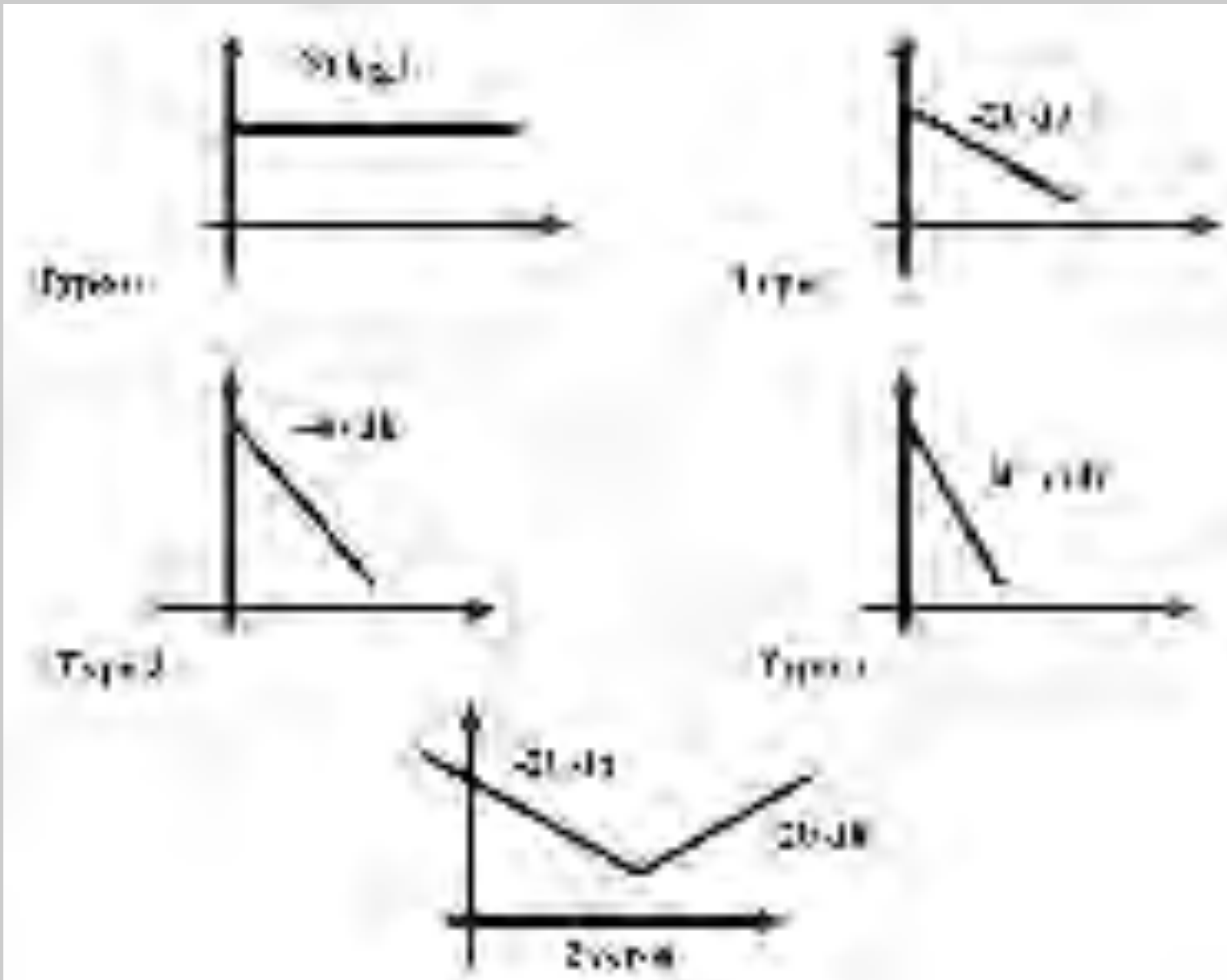
Identification by Bode Plot

- *Low frequency* indicate *the type* of the system.
- Intermediate frequency indicates the existence of zeros.
- If the poles and zeros are too close, it is so difficult to estimate accurately their locations.
- A *decade* is the frequency band from w to 10 w.



Identification by Bode Plot

- *Low frequency* indicate *the type* of the system.



Identification by Bode Plot

- The following table provides a few gains with the corresponding values in decibels.

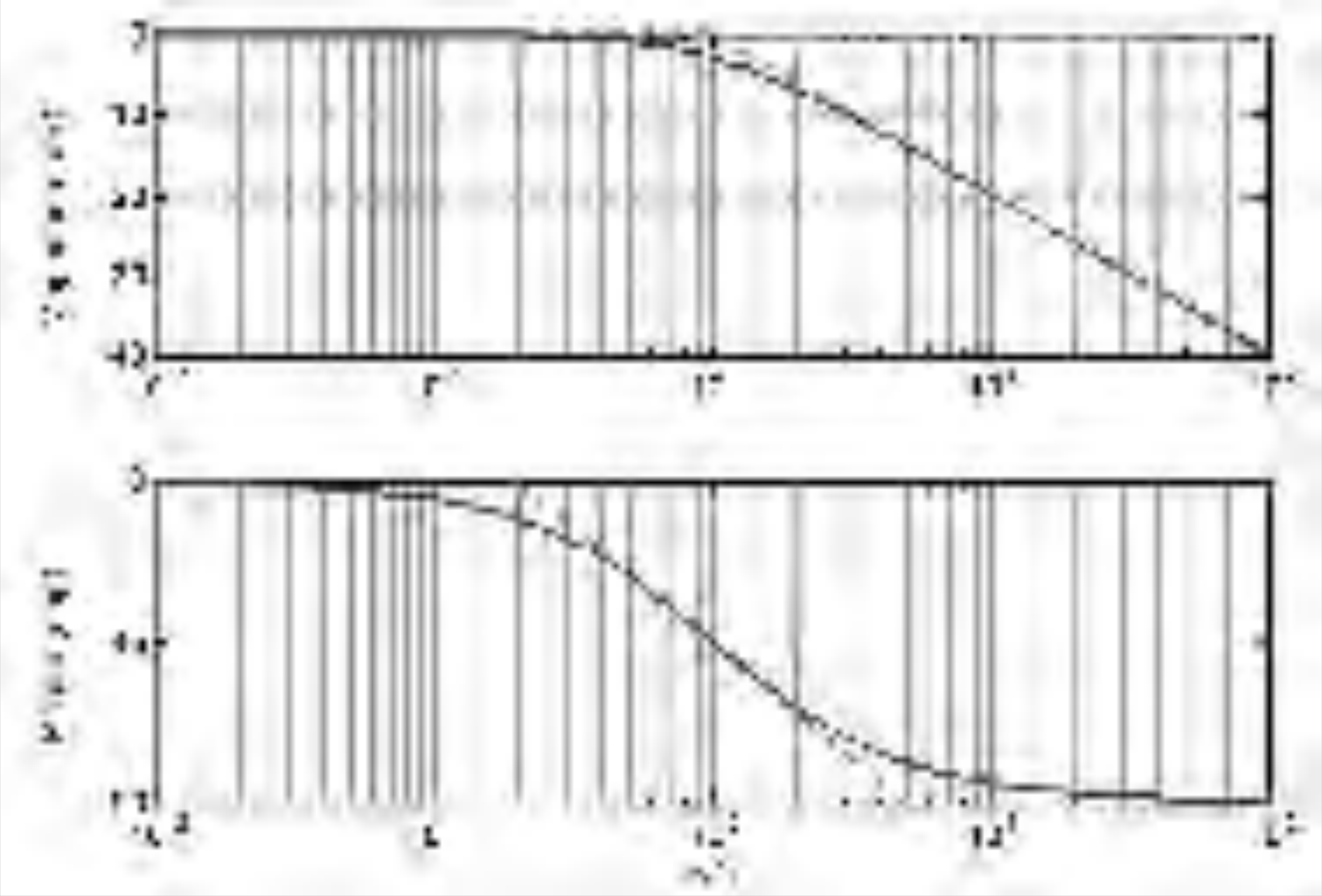
Magnitude	Decibels (dB)
0.001	-60
0.01	-40
0.1	-20
0.5	-6
$\frac{1}{\sqrt{2}}$	-3
1	0
$\sqrt{2}$	3
2	6
10	20
100	40
1000	60

Identification by Bode Plot

- If gain > 1 then +ve db.
- If gain < 1 then -ve db.
- If gain = 1 then 0 db.

Asymptotic properties of Bode Plot [6]

- Single pole $H(s) = \dfrac{a}{s+a}$

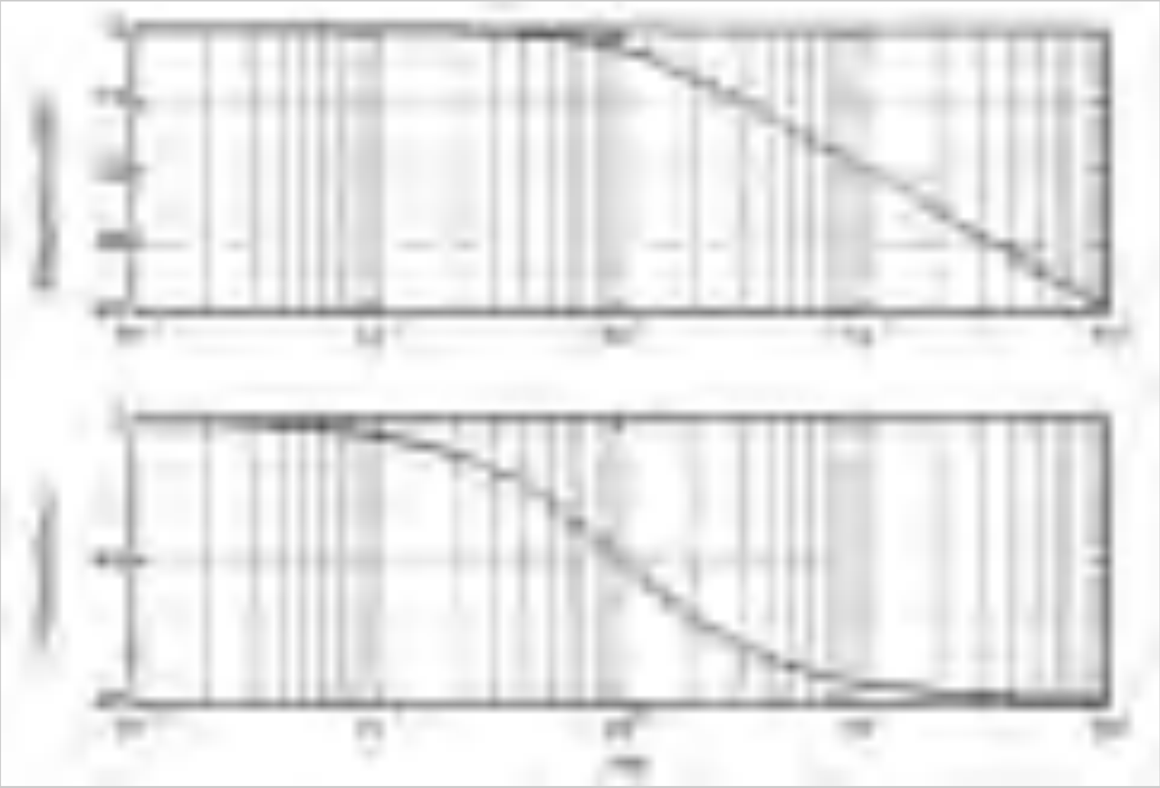


Asymptotic properties of Bode Plot

$$H(s) = \frac{a}{s+a}$$

Magnitude response

- Low-frequency asymptote (ω → 0), flat
- Breakpoint at ω = a
- High frequency asymptote, –20 dB/decade
- Actual curve is –3 dB below breakpoint

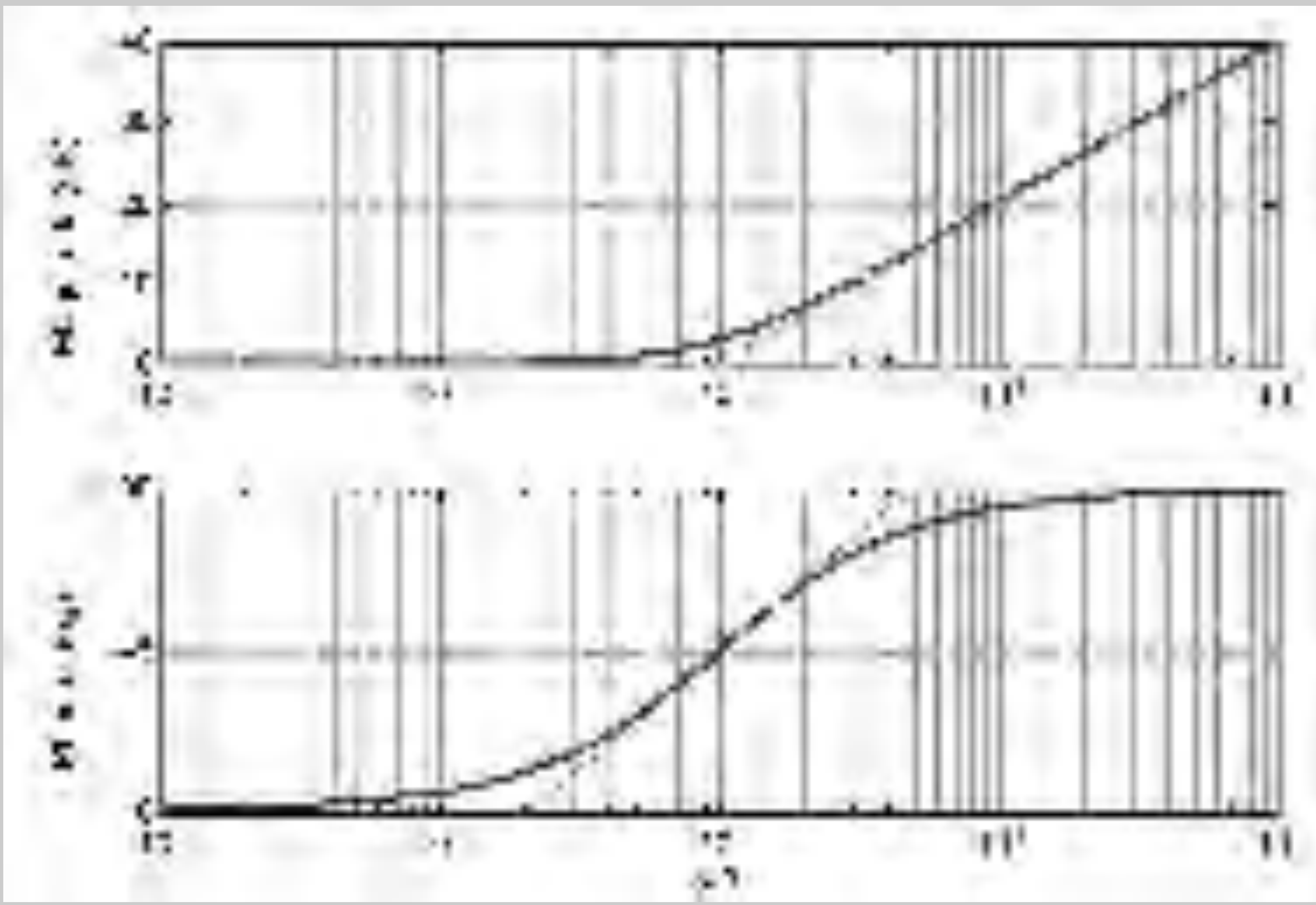


Phase response

- Low frequency asymptote = 0°
- –45° at breakpoint (ω = a)
- High frequency asymptote = –90°
- Central slope crosses 0° at ω ≈ a/5, –90° at ω ≈ 5a

Asymptotic properties of Bode Plot

- Single zero $H(s) = \frac{s+b}{b}$

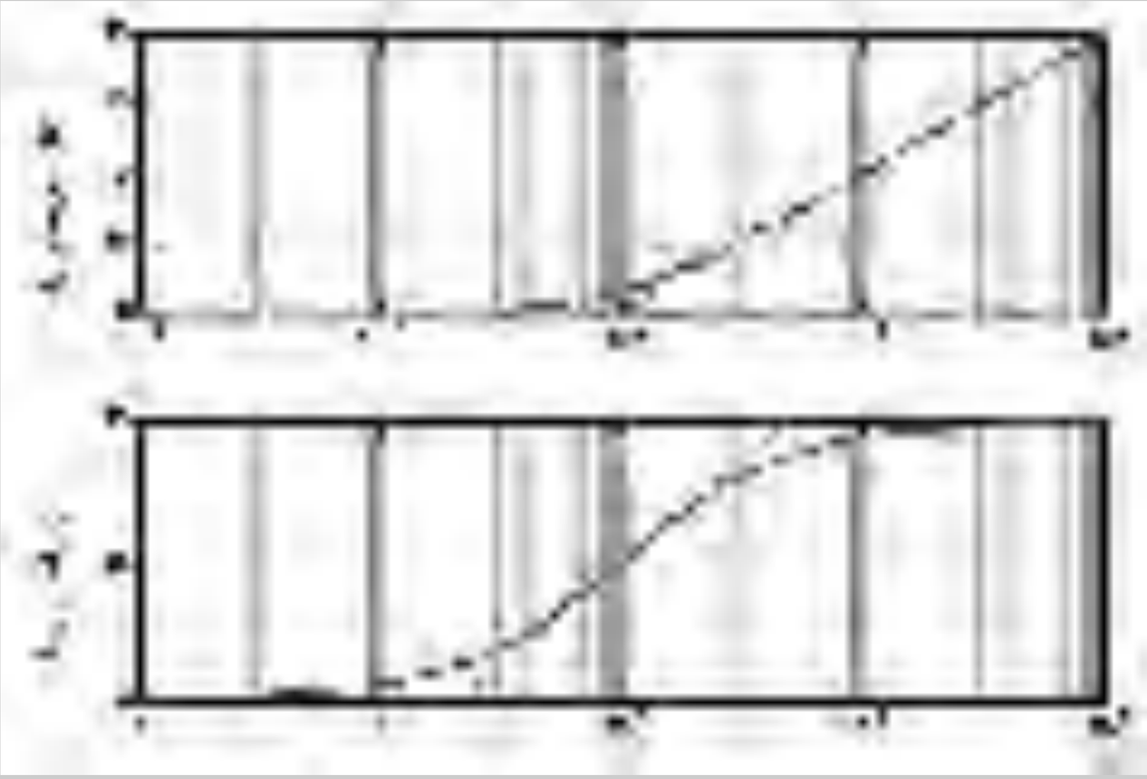


Asymptotic properties of Bode Plot

$$H(s) = \frac{s+b}{b}$$

Magnitude response

- Low-frequency asymptote ($\omega \to 0$), flat
- Breakpoint at $\omega = b$
- High frequency asymptote, +20 dB/decade
- Actual curve is +3 dB above breakpoint

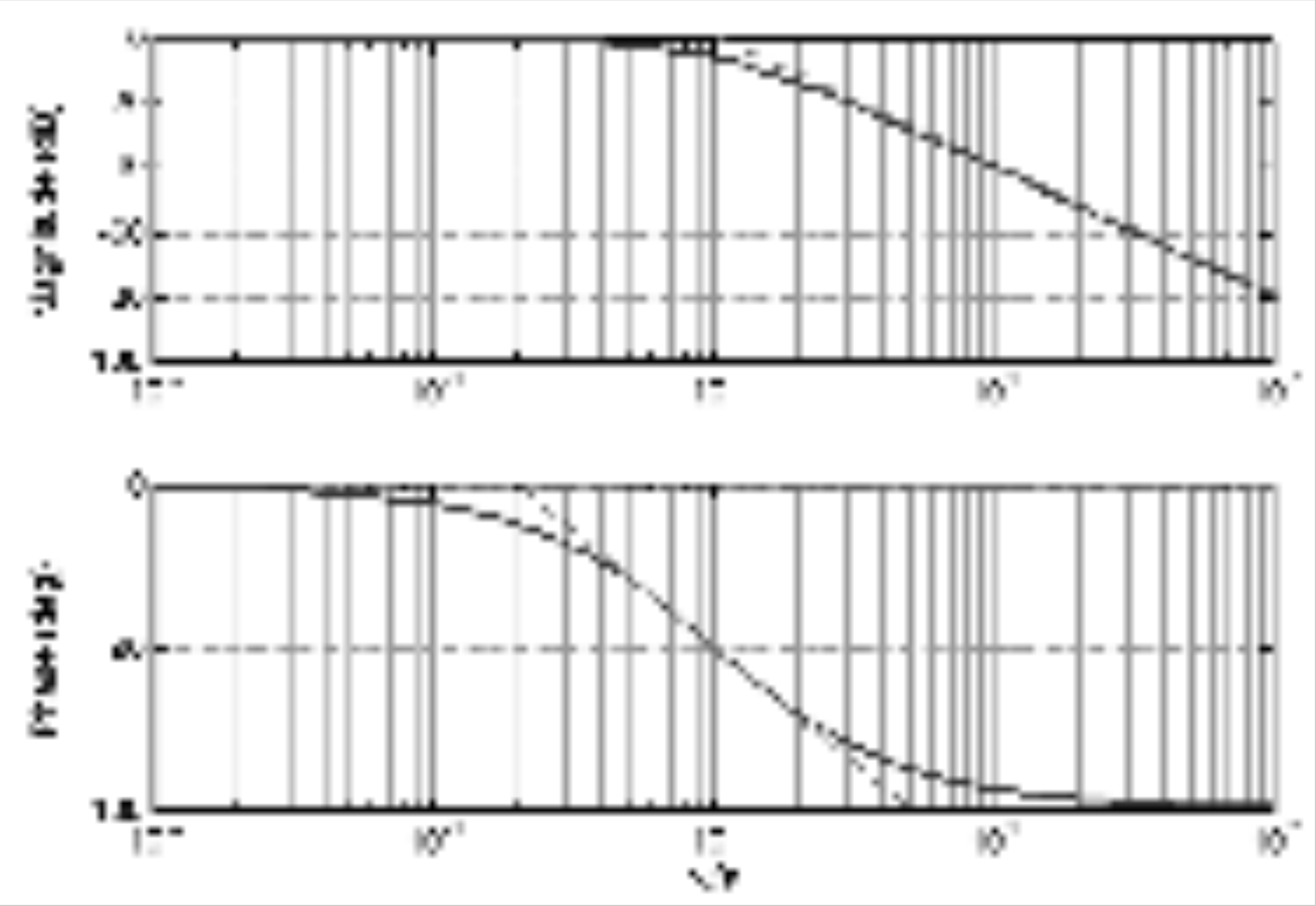


Phase response

- Low frequency asymptote = 0°
- +45° at breakpoint ($\omega = b$)
- High frequency asymptote = +90°
- Central slope crosses 0° at $\omega \approx b/5$, +90° at $\omega \approx 5b$

Asymptotic properties of Bode Plot

- Double pole, $H(s) = \dfrac{a^2}{(s+a)^2}$

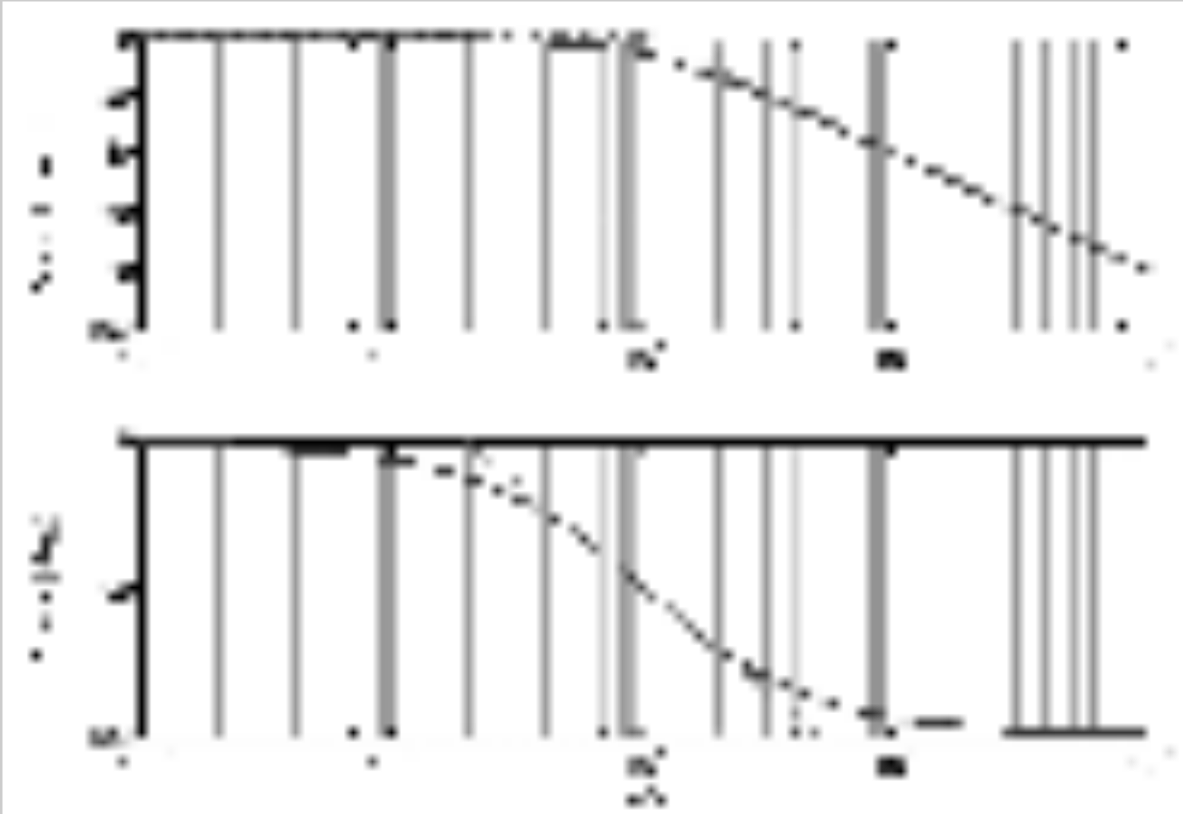


Asymptotic properties of Bode Plot

$$H(s) = \frac{a^2}{(s+a)^2}$$

Magnitude response

- Low-frequency asymptote ($\omega \to 0$), flat
- Breakpoint at $\omega = a$
- High frequency asymptote, –40 dB/decade
- Actual curve is –6 dB below breakpoint

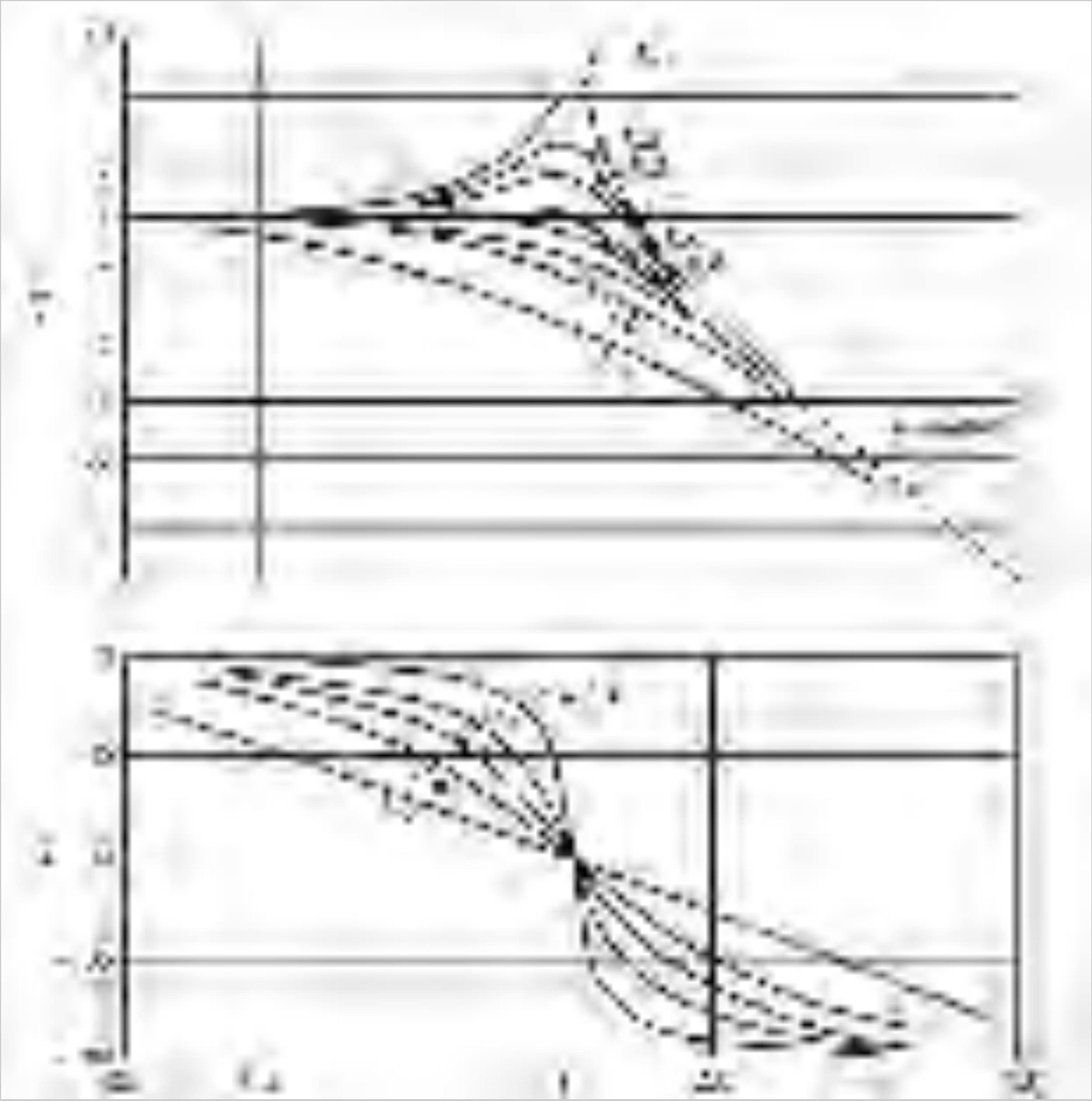


Phase response

- Low frequency asymptote = 0°
- –90° at breakpoint ($\omega = a$)
- High frequency asymptote = –180°
- Central slope crosses 0° at $\omega \approx a/5$, –180° at $\omega \approx 5a$

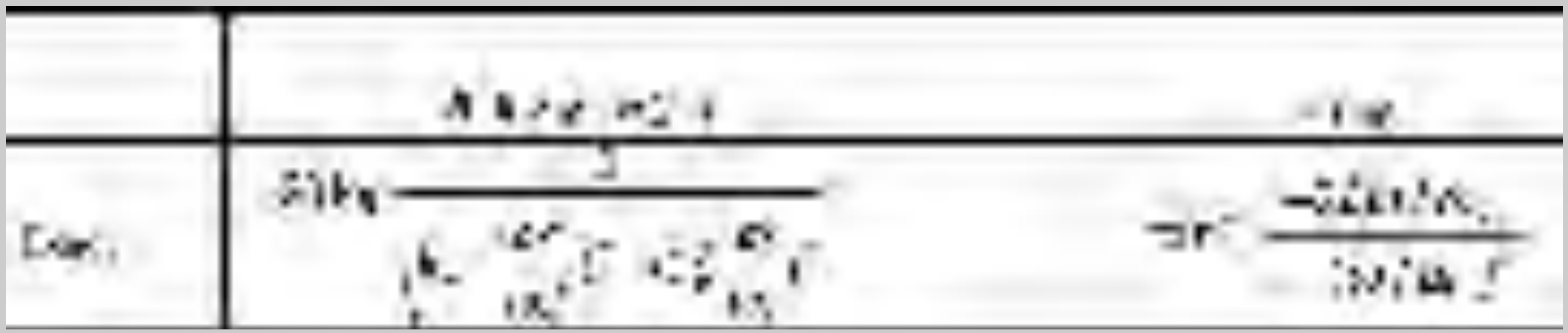
Asymptotic properties of Bode Plot

- Second order response $H(s) = \dfrac{\omega n^2}{s^2 + 2\zeta\omega n s + \omega n^2}$



Asymptotic properties of Bode Plot

$$H(s) = \frac{\omega n^2}{s^2 + 2\zeta\omega n s + \omega n^2}$$



- For $\zeta \geq 1$, the second order system is equivalent to two first-order systems in series.
- for $\zeta < 0.707$, the Magnitude curves attain maxima in the vicinity of $\omega/\omega n=1$.
- This can be checked by differentiating the expression for the magnitude with respect to $\omega/\omega n$ and setting the derivative to zero.

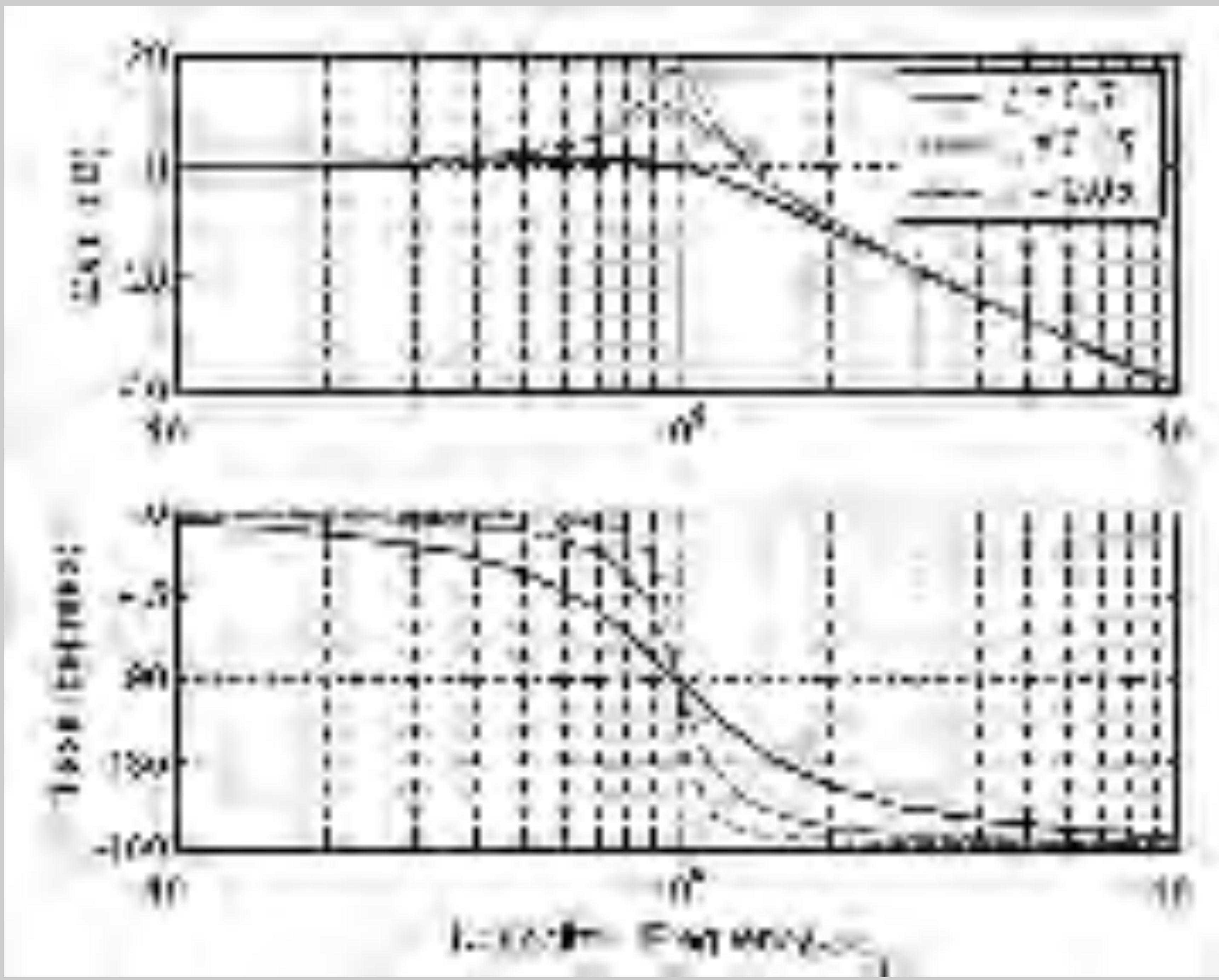
Asymptotic properties of Bode Plot

$$H(s) = \frac{\omega n^2}{s^2 + 2\zeta\omega n s + \omega n^2}$$

	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

Asymptotic properties of Bode Plot

- Second order underdamped response $H(s)=\dfrac{\omega n^2}{s^2+2\zeta\omega ns+\omega n^2}$



Bode Plot properties of underdamped 2d order System

Magnitude response

$$H(s) = \frac{\omega n^2}{s^2 + 2\zeta\omega n s + \omega n^2}$$

- Low-frequency asymptote ($\omega \to 0$), flat
- Breakpoint at $\omega = \omega n$
- High frequency asymptote, –40 dB/decade
- is at height $1/(2\zeta)$
- When $\zeta < 0.707$, the actual maximum occurs at $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$,

  and the actual maximum (Resonant peak) value is $\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$
- For sufficiently small ζ, this point coincides with ωn and $1/(2\zeta)$.

Asymptotic properties of Bode Plot

$$H(s) = \frac{\omega n^2}{s^2 + 2\zeta\omega n s + \omega n^2}$$

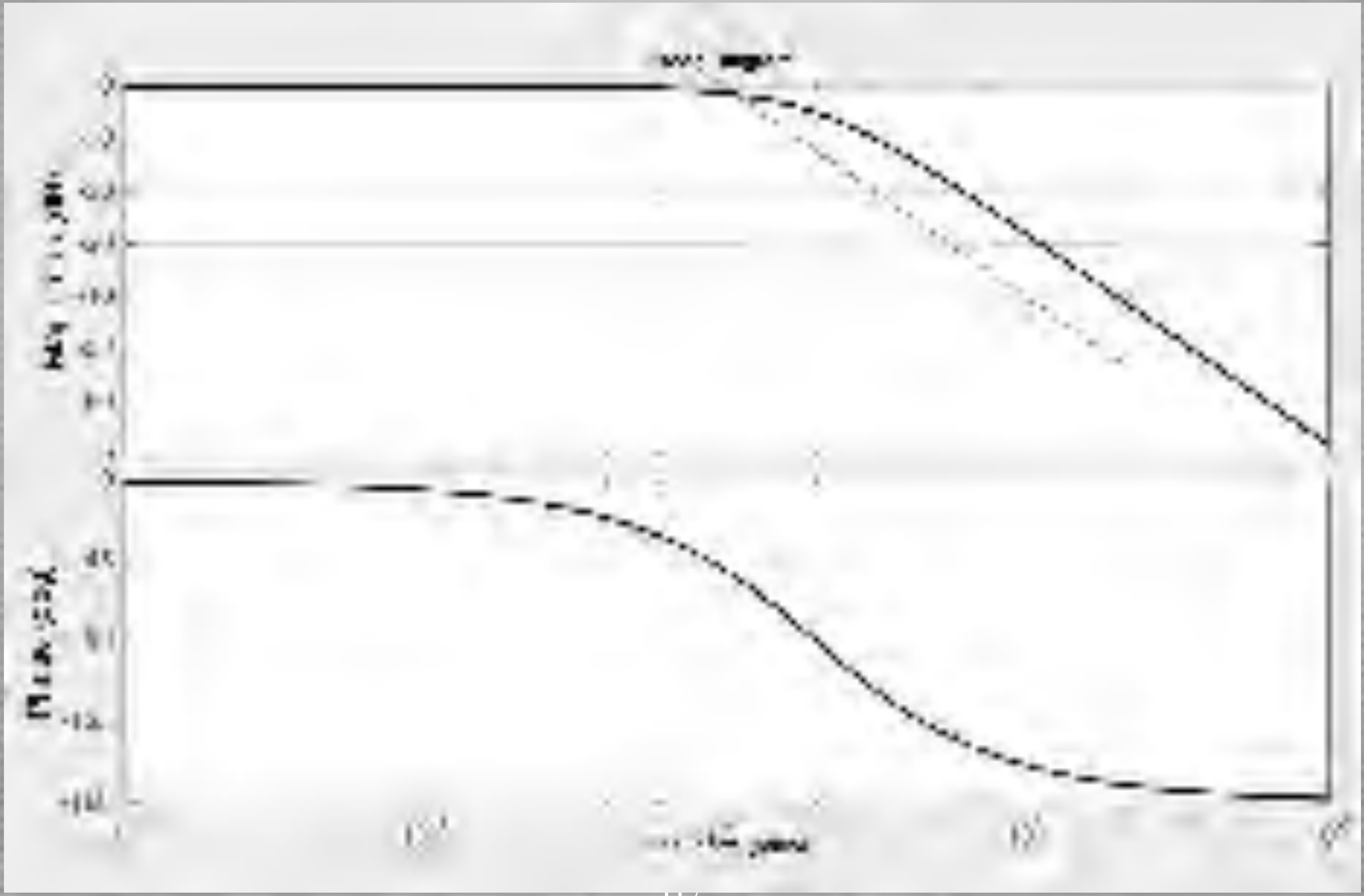
Phase response

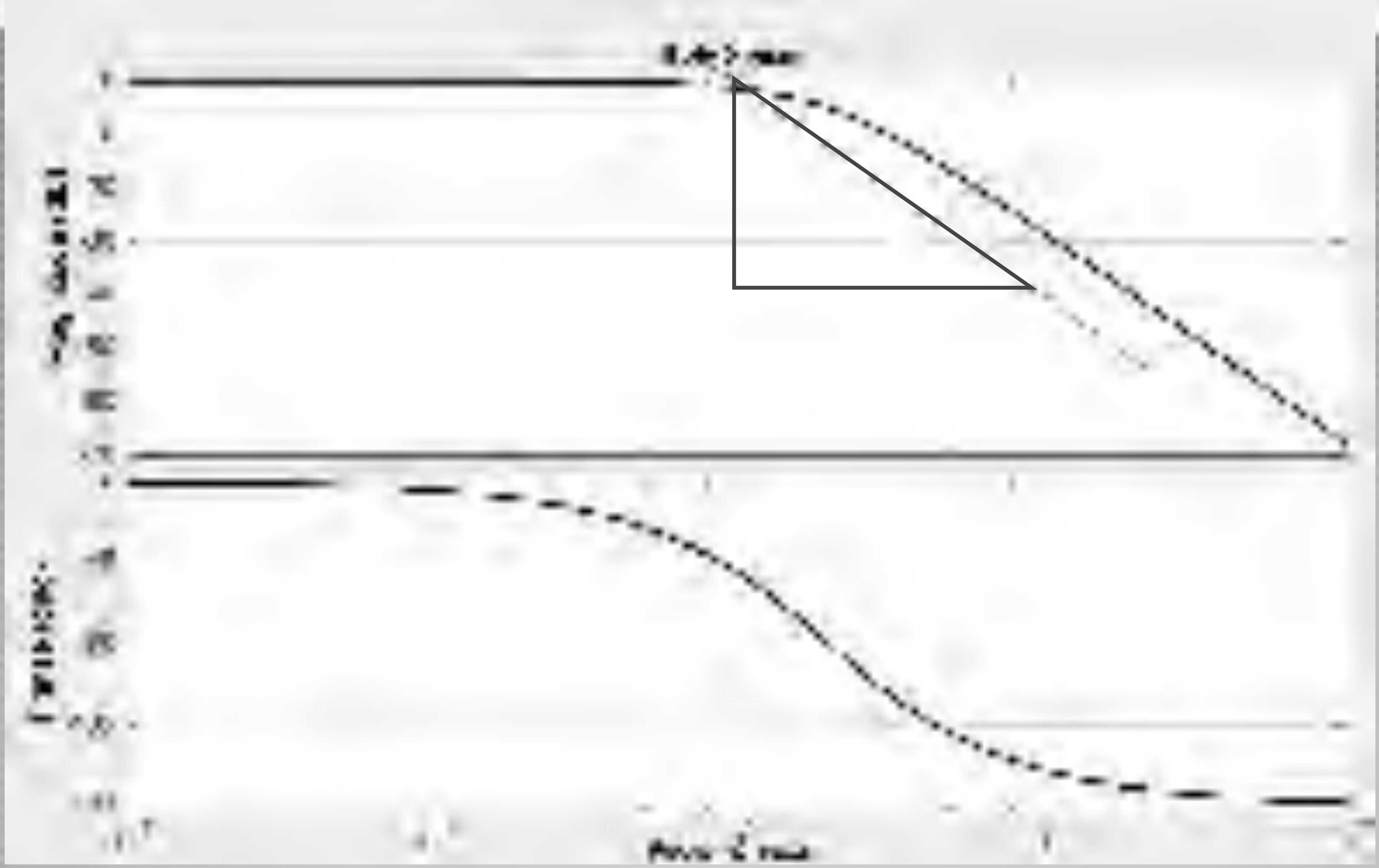
- Low frequency asymptote = 0°
- –90° at breakpoint (ω = a)
- High frequency asymptote = –180°
- Central slope crosses 0° at $\omega \approx \frac{\omega_n}{5^\zeta}, \qquad -180 \;\; at \quad \omega \approx \omega_n 5^\zeta$

ID from Bode plot

Example

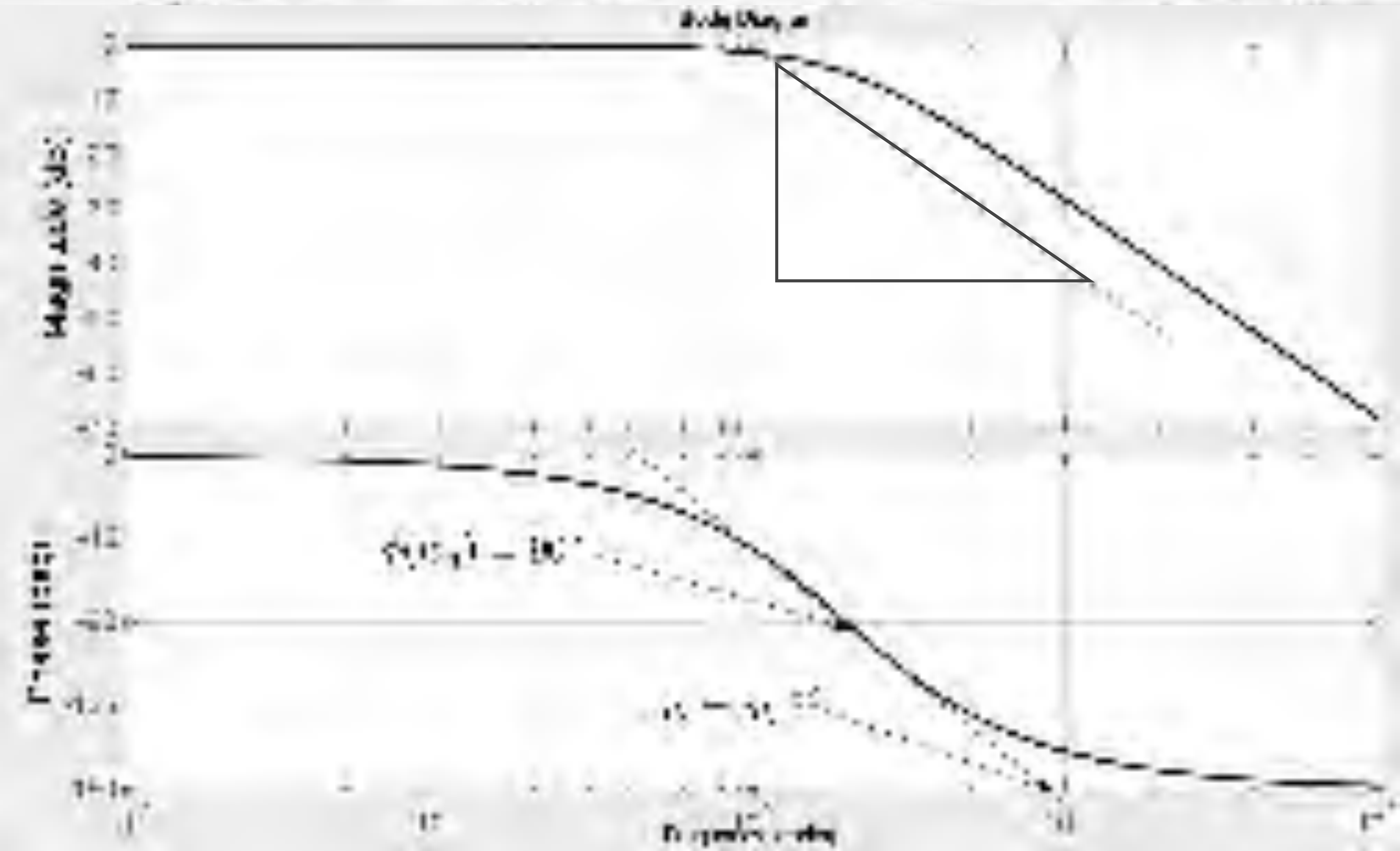
Example1: using the following Bode plot, find the system model and identify its parameters.





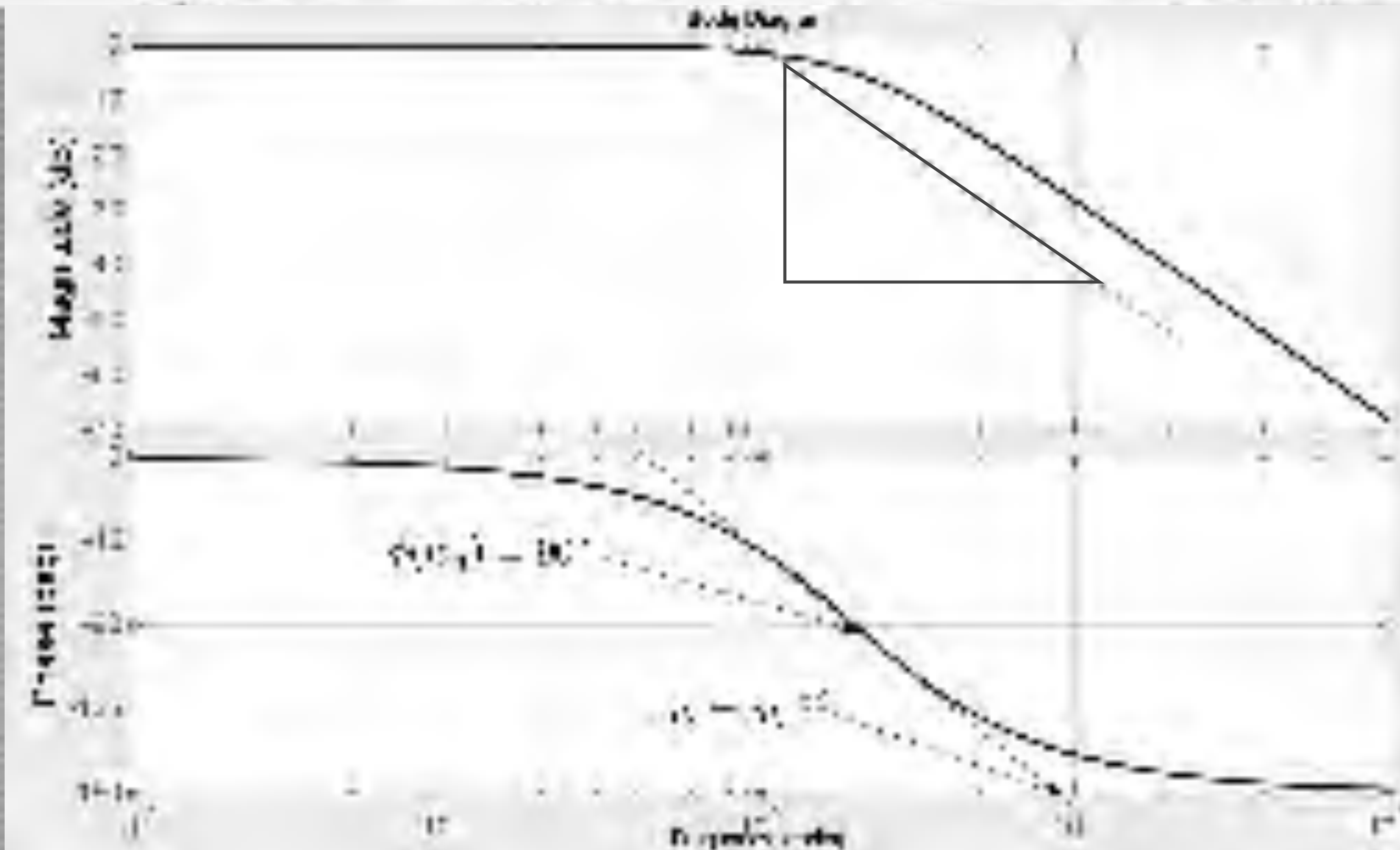
1. From Mag. plot: High frequency asymptote, −40 dB/decade with flat low frequency mag.
2. From phase plot: the break frequency is at $\phi(\omega_n) = -90^o$, hence the model is

$$H(s) = \frac{\omega n^2}{s^2 + 2\zeta\omega ns + \omega n^2}$$



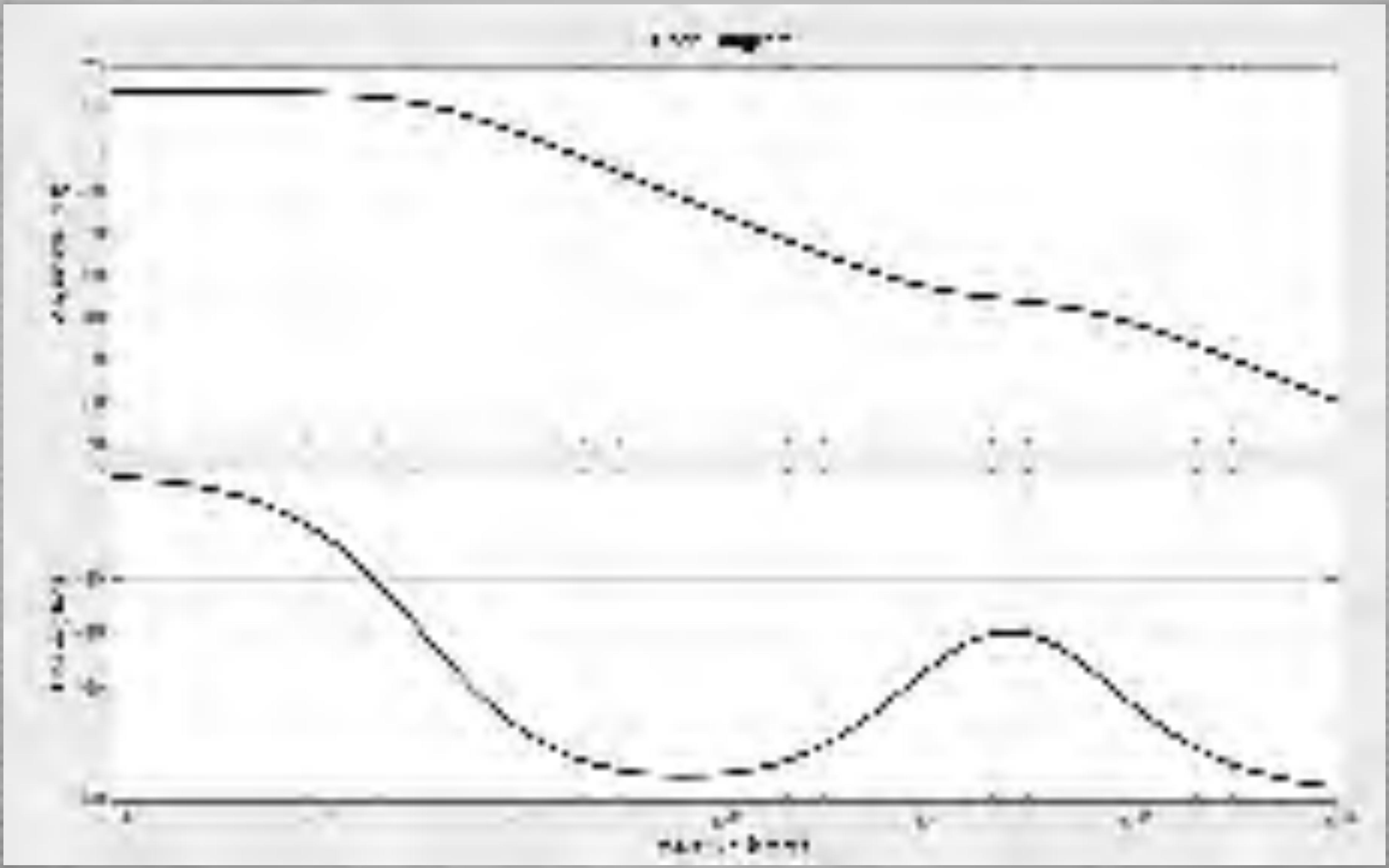
From phase plot, $\omega_n = 2rad/\sec.$, the central slope crosses

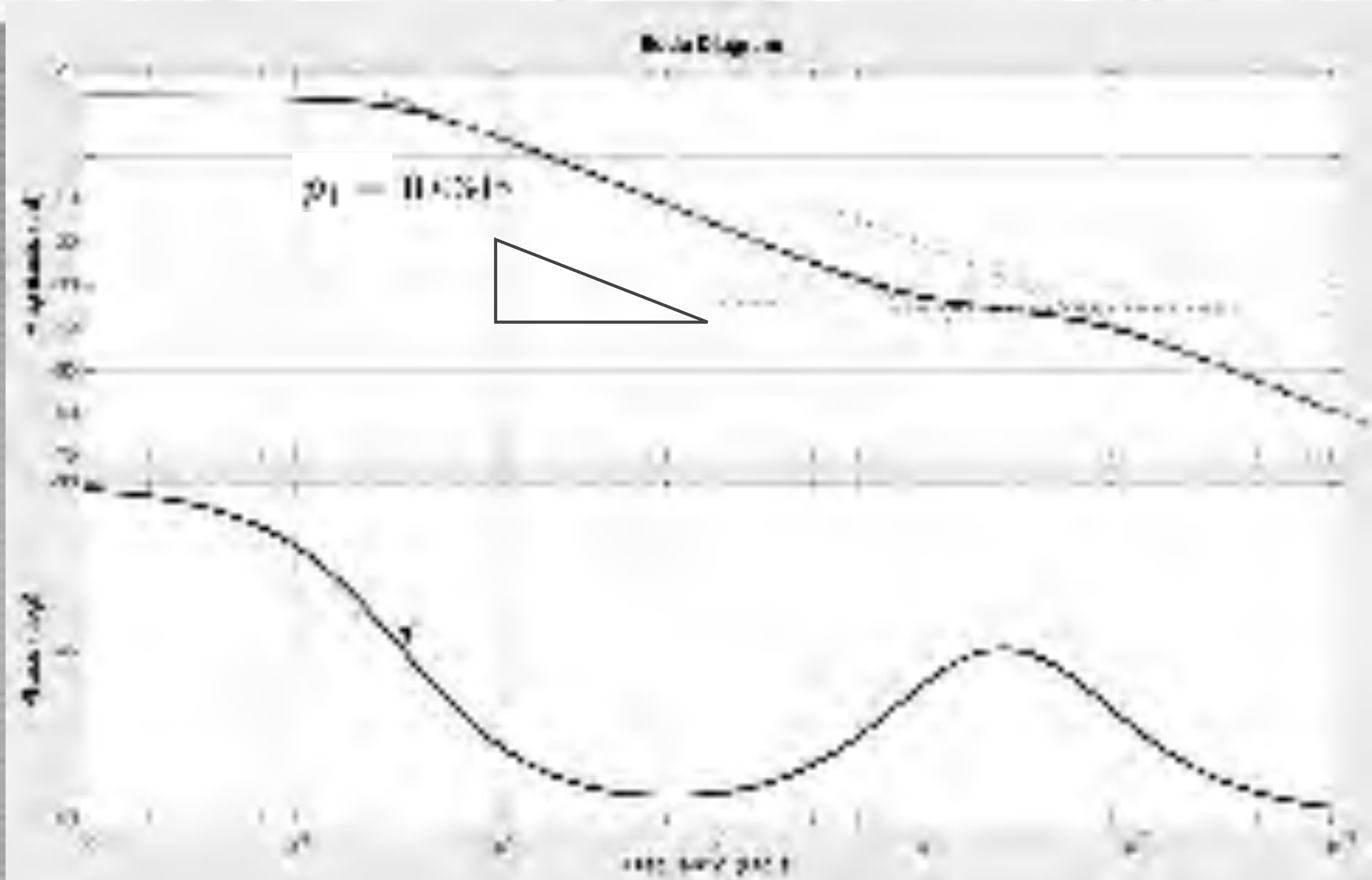
$$\phi = -180^o \, at \quad \omega = 9rad/\sec. \quad \Rightarrow 9 = 2\times 5^{\zeta} \Rightarrow \; 4.5 = 5^{\zeta} \Rightarrow \zeta = 0.935$$



$$H(s) = \frac{4}{s^2 + 3.6s + 4}$$

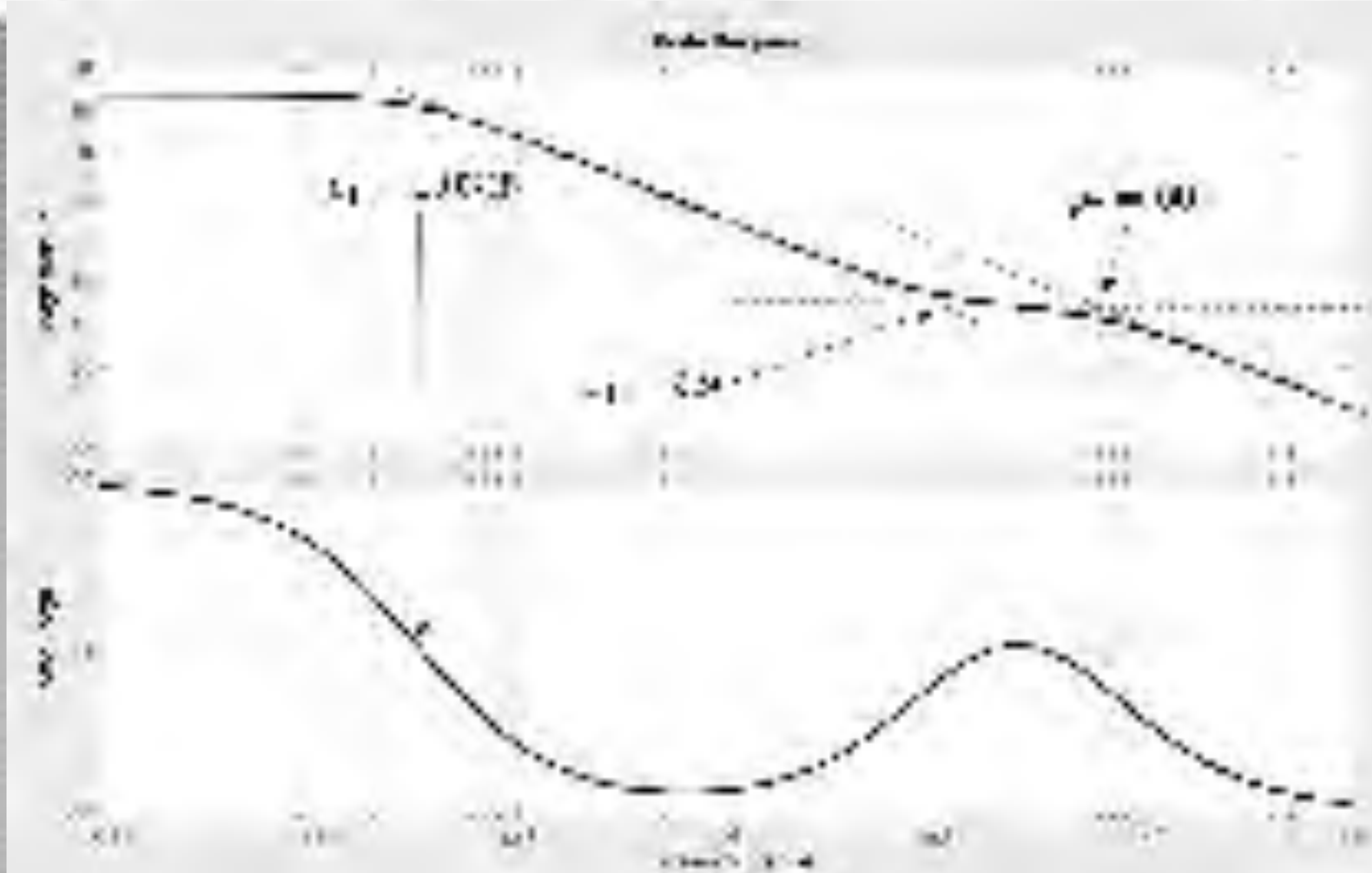
Example2: using the following Bode plot, find the system model and identify its parameters.





From Mag. & phase plots, we have two simple poles and a simple zero:

$$H(s) = \frac{K(s/z_1 + 1)}{(s/p_1 + 1)(s/p_2 + 1)}$$



To find K,

$$20\log|H(j\omega)|_{\omega=0} = 13.9dB, \qquad 20\log(K) = 13.9dB \quad ,K = 4.955$$

$$H(s) = 4.9545\frac{(s/9.9+1)}{(s/0.034+1)(s/60+1)} = \frac{10(0.1s+1)}{s^2+60.03s+2.04}$$

- [1]: L. Ljung, “System Identification: Theory for the User”, Prentice Hall PTR, New Jersey, USA, 1999.
- [2]: E. Ikonen and K. Najem, “Advanced Process Identification and Control”, Marcel Dekker, Newyork, USA, 2002.
- [3]: J. Mikles and M. Fikar, “Process Modelling, Identification and Control”, Springer, Berlin, 2007.
- [4]: http://www.users.abo.fi/khaggblo/PDC/PDC5milver.pdf
- [5]:D. R. Coughanowr and S. E. LeBlanc, “Process SystemsAnalysis and Control”, McGraw-Hill Companies, Inc., 2009.
- [6]: http://www.dartmouth.edu/~sullivan/22files/Bode_plots.pdf

System Identification

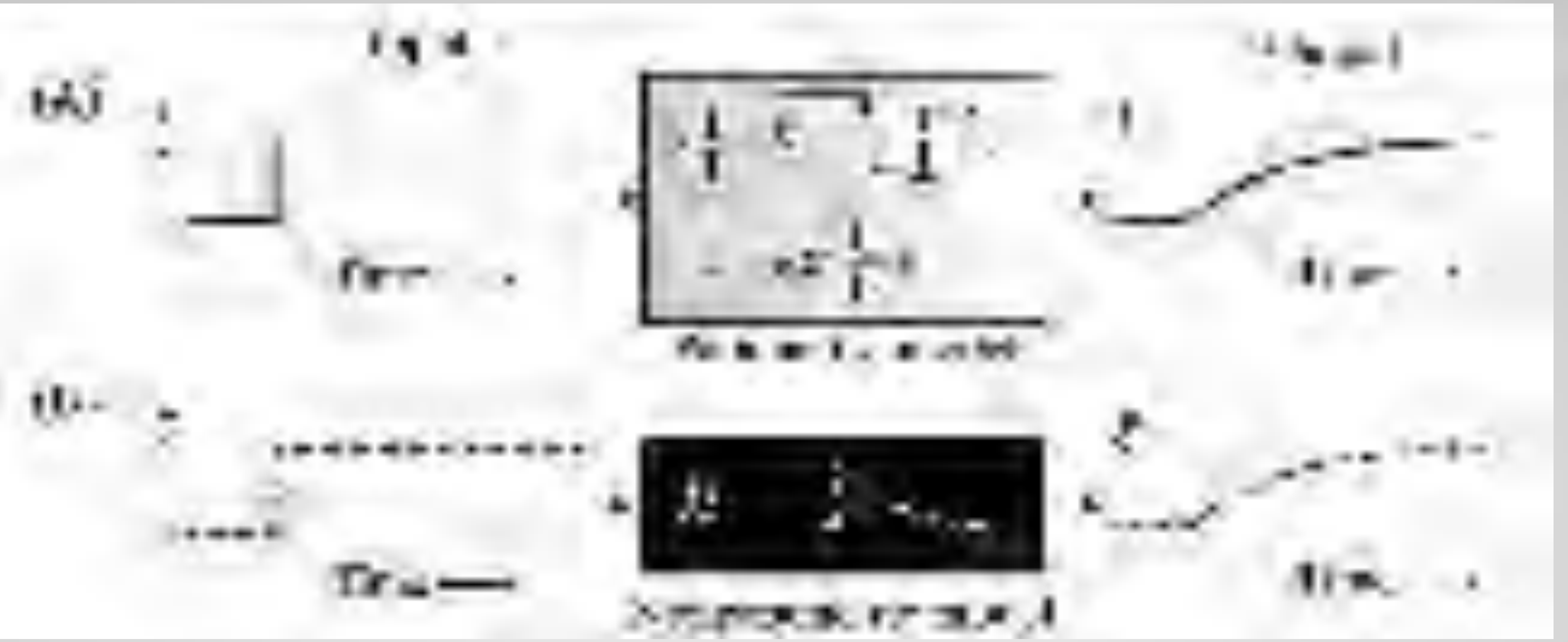
A Third-year Course for Control and Mechatronics Engineering

By Dr. Taghreed M. MohammadRidha

Non-parametric Identification

- Nonparametric identification techniques provide a very effective and simple way of finding model structure in data sets without the imposition of a parametric one.
- Commonly, the initial process to carry out is the nonparametric identification
- If it were suitable, the parametric identification should be performed

Parametric vs. Non-parametric model



(B) The black box, nonparametric equivalent of the same system is the white curve representing the (sampled) unit impulse response (UIR).

- Convolution of the input time series *x(t)* with the system's UIR *h(t)* generates the system's output time series *y(t)*:

$$y(t) = h(t) \otimes x(t)$$

Useful Definitions & Tools

- Discrete-time signals

A discrete-time signal $y(k) := y_c\ (kT_s)$ is the sampling of the continuous-time signal y_c

- T_s is the sampling period.
- k is an integer running index: k = 1, 2, … .
- The sampling frequency ωs is defined by $\omega s = 2\pi/T_s$

Useful Tools & Recalls

- Discrete-time models are represented by *difference equations*:

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_1 u(k) + \cdots + b_m u(k-m)$$

Or

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{j=1}^{m} b_j u(k-j)$$

Recalls: Energy and Power

- In signal processing, total **energy** of:

-A <u>Continues</u> signal x(t):

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \, .$$

-A <u>Discrete</u> signal x(n):

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \, .$$

- The signal **power** in

-A Continues signal x(t) is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \, .$$

-A Discrete signal x(n) is

$$P = \lim_{N \to \infty} \left(\frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \right) .$$

Recalls: Energy and Power

- A signal can be categorized into energy signal or power signal.
- An **energy signal** has a finite energy, $0 < E < \infty$. (e.g. exponential decay).
- The power of an energy signal is 0.
- On the contrary, the **power signal** is not limited in time. (e. g. sine wave).
- The energy of a power signal is infinite

[illegible]

2) $x(n) = u(n)$ unit step: $\mathrm{P} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} u(n)^2 = \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$.

Useful Tools: Random Variables

- Continuous random variables are random quantities that are measured on a continuous scale. Typically random variables that represent, for example, time or distance will be continuous rather than discrete.
- Discrete random variables can take on only a sequence of values, usually integers. For example - Number of broken eggs in a batch or the number of bits in error in a transmitted message.
- Random variables are usually denoted by capital letters X. The values of the variables are usually denoted by lower case letters x.

Useful Tools & Recalls

- A stochastic system: systems in which the time variables change randomly.
- do not always produce the same output for a given input.
- A few components of systems (that can be stochastic in nature) include:

  stochastic inputs, random time-delays, noisy (modelled as random) disturbances, and even stochastic dynamic processes.
- The variables can be characterized by a probability function (i.e. they are statistically related).

Recalls: stochastic system

- **Wide-sense Stationary process** is a stochastic process whose mean function and its correlation function do not change by time shifts.
- A process is **ergodic** : if its statistical properties can be deduced from a single, sufficiently long, random sample of the process.

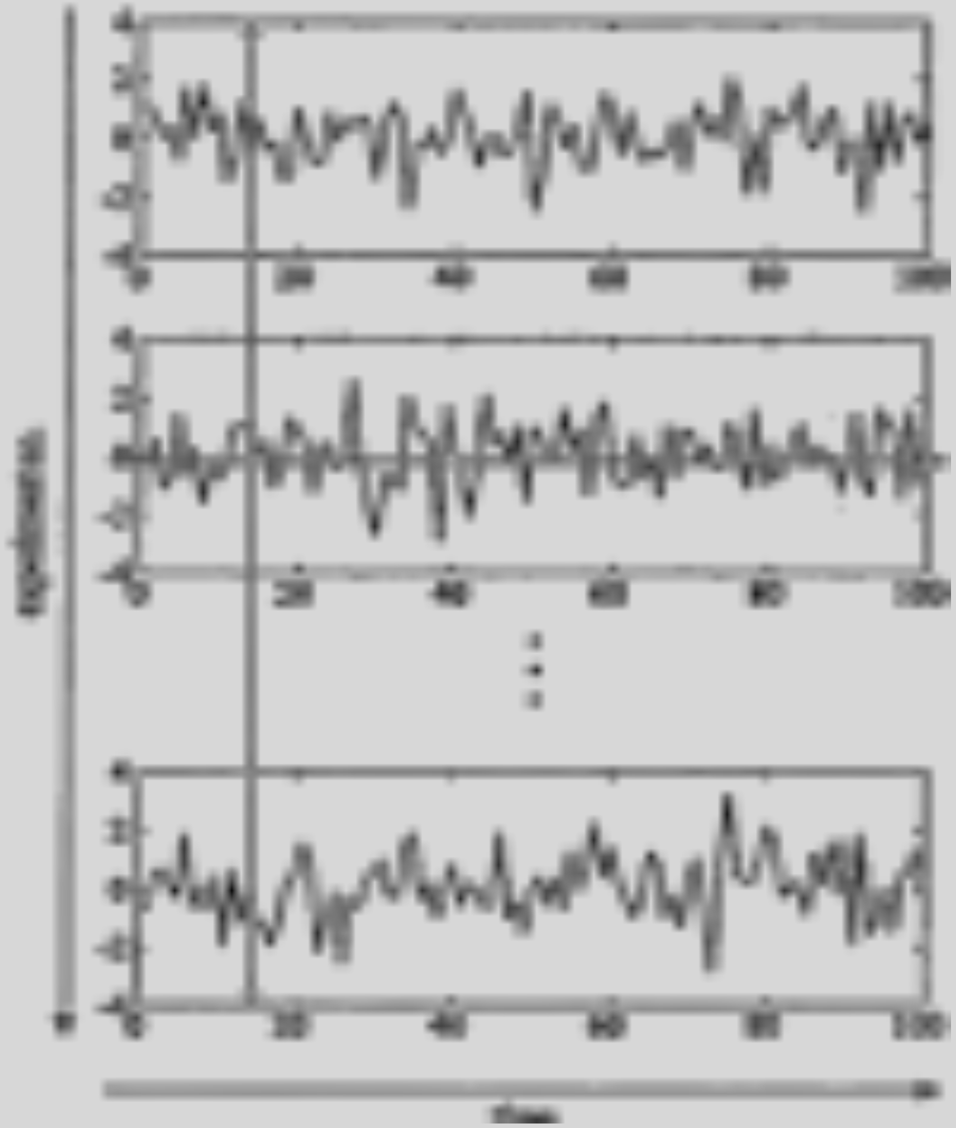
Stationarity & Ergodicity [2]

A stochastic process is called "stationary" if its statistical properties do not depend on time, i.e., if

$$f(x,k_1) = f(x,k_2) \quad \text{for all } k_1 \text{ and } k_2$$

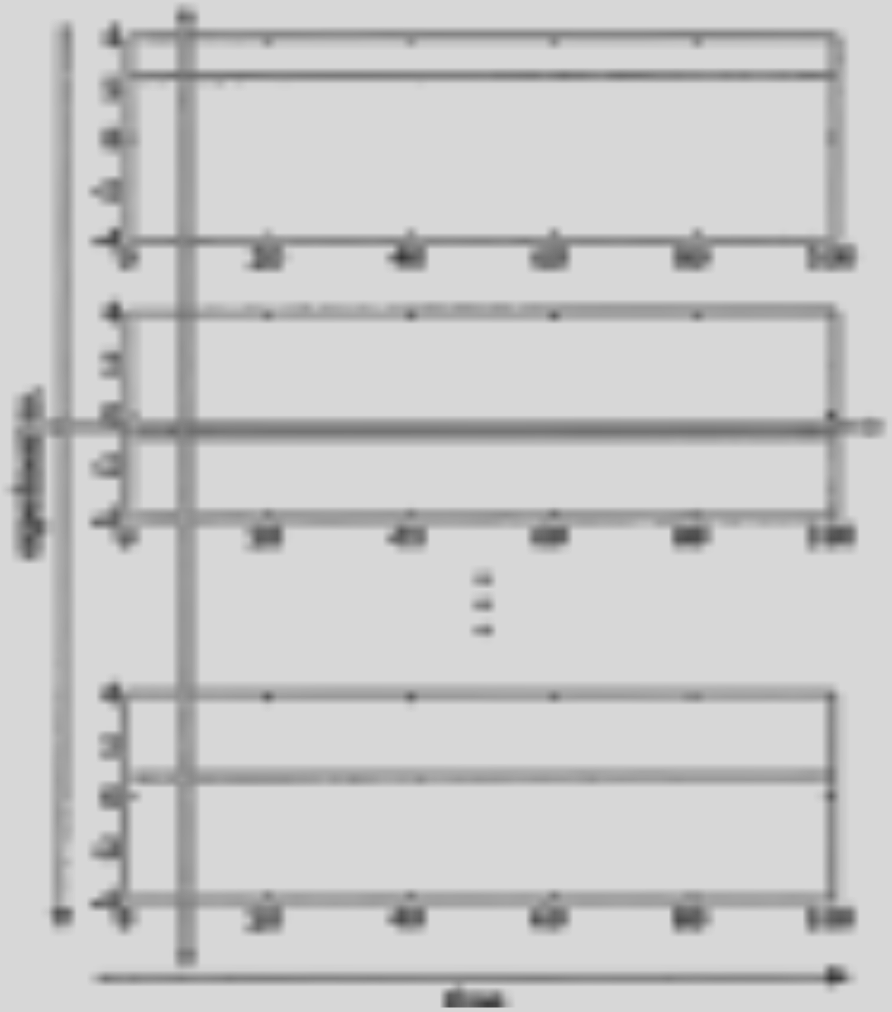
A stochastic process is called "ergodic" if the expectation over its realizations can be calculated as the time average of one realization.

An ergodic process is stationary.



Stationarity & Ergodicity [2]

A stationary process but non-ergodic.



It is stationary since the statistical properties of this process do not depend on time.

It is non-ergodic since one realization does not reveal the statistical properties of the process.

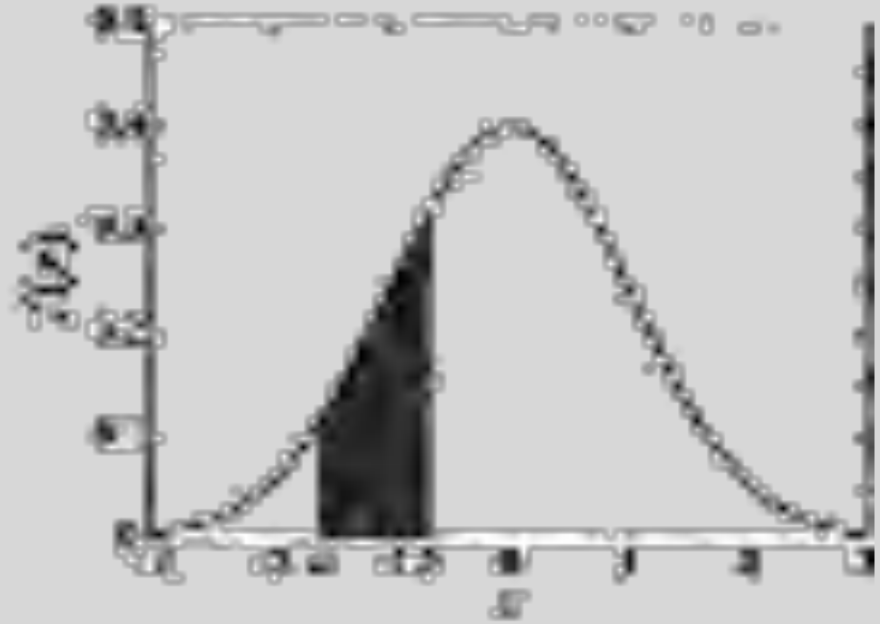
Useful Tools: Statistical

- Density Function [1] f(x) describes the probability distribution of a continuous random variable X. It has the following properties
- f(x)≥0 for all x.
- $\int_{-\infty}^{\infty} f(x)dx = 1.$
- A random variable X can be obtained by picking a point at random from under the density curve f(x) and then reading off the x-coordinate of that point.
- If X is a discrete random variable then f(x) is the Probability function and $\sum_x f(x) = \sum_x P(X = x) = 1.$

Probability Density Function

A Gaussian pdf $f(x)$ for a random variable x (Normal distribution)

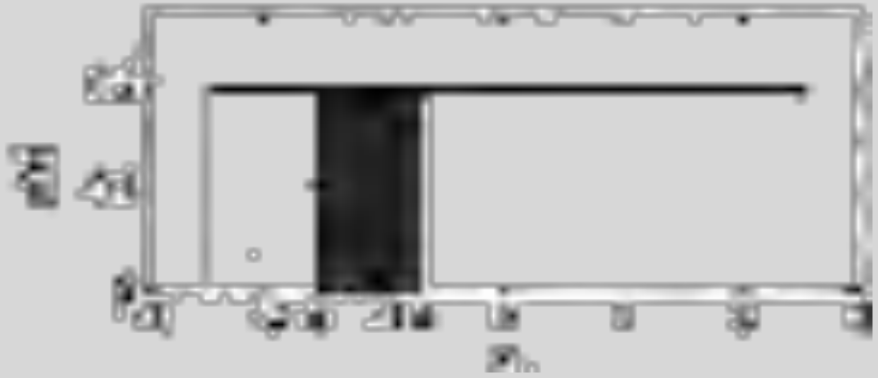
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{(-0.5x^2)}$$



$$P(a < x < b) = \int_a^b f(x)dx$$

Uniform distribution

$$p(x) = \begin{cases} 0.2 & 2.5 < x < 2.5 \\ 0 & \text{other values} \end{cases}$$



$$P(a \le x \le b) = \int_a^b p(x)dx$$

Statistical: Density

- Example1: Suppose the income (in tens of thousands of dollars) of people in a community can be approximated by a continuous distribution with density

$$f(x) = \begin{cases} 2x^{-2} & if\ x \geq 2 \\ 0 & if\ x < 2 \end{cases}$$

a) Find the probability that a randomly chosen person has an income between \$30, 000 and \$50,000.

b) Find the probability that a randomly chosen person has an income of at least \$60,000.

Statistical: Density

- Example 1:

- (a)Sol. Let X be the income of a randomly chosen person. The probability that a randomly chosen person has an income between \$30,000 and \$50,000 is

[illegible]

Useful Tools: Statistical

- Mean or Expected value[1] $\mathrm{E}(x)$ of a random variable x is the long-run *average* value of repetitions of the experiment it represents.
- The arithmetic MEAN of the values converges to $\mathrm{E}(x)$ as the number of repetitions approaches infinity.
- The expected value is also known as the **expectation**, **average**, **mean value.**

Statistical: Expected Value

- For a continuous random variable, the **mean** μ_X of a continuous random variable X with probability density function $f(x)$ is

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x\, f(x) dx\,, \qquad f(x) \geq 0$$

- For a discrete random variable the expected value μ_X of a discrete random variable X with probability function $P(x)$ is

$$\mu_X = E[X] = \sum_{x} x\ P(x), \qquad P(x) \geq 0$$

Statistical: Expected Value

- If we have two random signals or variables, their averages can reveal how the two signals interact.
- If we have a random process in which only one sample can be viewed at a time, then we will often not have all the information available to calculate the mean using the density function as shown above.

Statistical: Expected Value

- When we can not view the entire ensemble of the random process, **or when f(x), P(x) are unknown**, we must use time average.
- The time averages will also only be taken over a finite interval T since we will only be able to see a finite part of the sample.
- Generally, this will only give us acceptable results for independent and **ergodic** process.
- The mean value of an ergodic random process can be estimated by

$$\bar{X} = \frac{1}{T} \int_0^T X(t)\ \mathrm{d}t$$

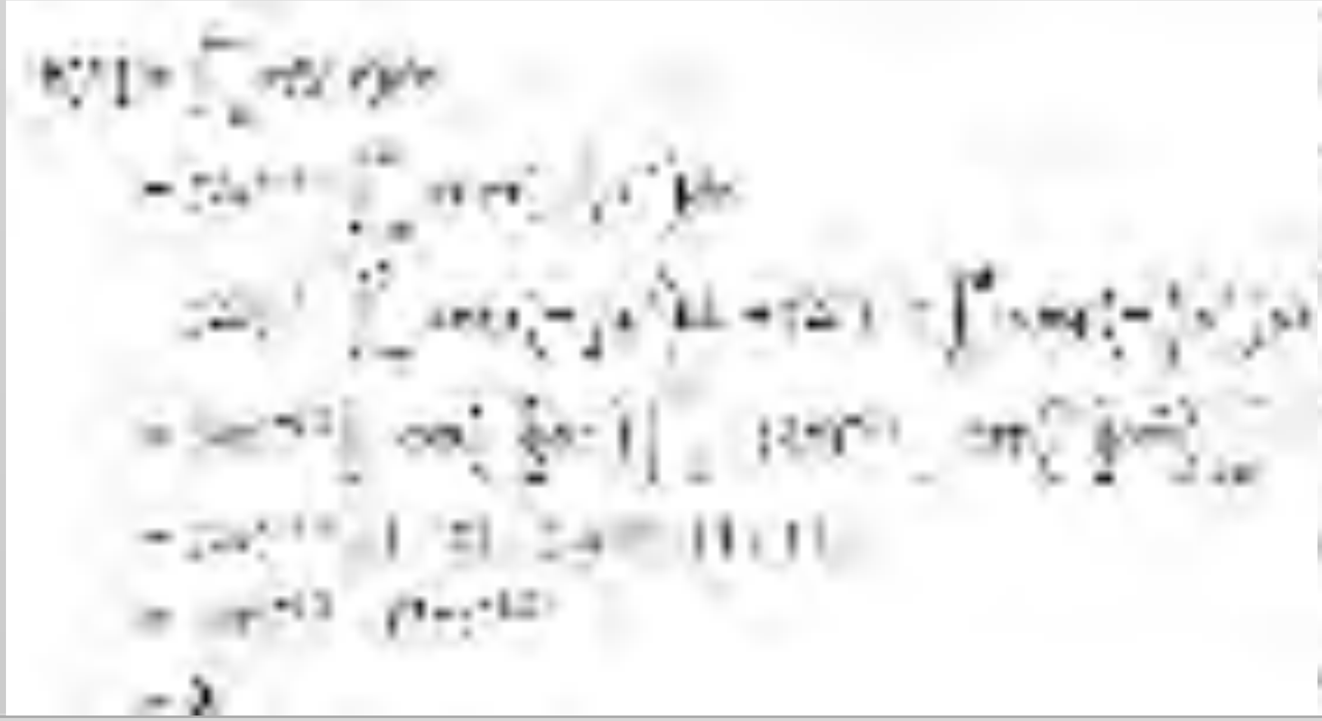
- For discrete ergodic process the mean is estimated as $\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X[i]$

Statistical: Expected Value

- Example 3: What is the expected value of the continuoes random variable X which is normally distributed, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{(-0.5x^2)}$$

- Sol.



Statistical: Expected Value

- Example 1: X is a discrete random variable, the table below defines a probability distribution for X. What is the expected value of X?

x	$P\{X = x\}$
−40	0.12
−30	0.04
−20	0.05
−10	0.17
0	0.02

- Sol.
- $E[X] = \sum xP(X)$
- $E[X] = (-40)(0.12) + (-30)(0.04) + (-20)(0.05) + (-10)(0.17)$
- $E[X] = -8.7.$

Statistical: Expected Value

- Example 2: You toss a coin until a tail comes up. $P(x) = 1/2^x$. What is E[X]?
- Sol.
- Insert your "x" values into the first few values for the formula, one by one: $E[X]=1/2^0 + 1/2^1 + 1/2^2 + 1/2^3 + 1/2^4 + 1/2^5$.
- $= 1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 1.96875$.
- **Note:** What you are looking for here is a number that the series converges on.
- In this case, **it converges to 2, so that is your EV.**
- The function must stop at a particular value. If it doesn't converge, then there is no Expected Value.

Statistical: The Variance

- The variance is a measure of spread data around their means.

$$\mathrm{Var}(X) = \sigma^2 = E[\quad (X - E[X])^2 \quad]$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{X})^2 f(x) dx$$

- This can be rewritten as:

$$\sigma^2 = \overline{X^2} - (\bar{X})^2$$

$$\sigma^2 = E[X^2] - (E[X])^2$$

System Identification

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Statistical: The Standard Deviation

- The most common way to describe the range of variation *is standard deviation* (**σ**).
- The standard deviation is simply the ***positive square root*** of ***the variance***.

…____________________…

Statistical: The Standard Deviation

- Example [1]: Suppose a train arrives shortly after 1:00 PM each day, and that the number of minutes after 1:00 that the train arrives can be modeled as a continuous random variable with density

$$f(x) = \begin{cases} 2(1-x) & if \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and standard deviation of the number of minutes after 1:00 that the train arrives.

Statistical: The Standard Deviation

- Example [1]: cntd..

Statistical: Correlation & Covariance [2]

- Mean and variance are defined for single random variable or stochastic process.
- In contrast **correlation and covariance** are defined for two random variables or stochastic processes.
- Both **correlation and covariance**, measure the similarity between two *random variables or stochastic processes.*

$$Corr(X,Y) = E[X.Y]$$

$$Cov(X,Y) = E\left[(X - \mu_x).\left(Y - \mu_y\right)\right]$$

Statistical: Correlation & Covariance [2]

$$Corr(X,Y) = E[X.Y]$$

$$Cov(X,Y) = E[(X - E[X]).(Y - E[Y])]$$

- **If** $\mathrm{E}[X] = E[Y] = 0,$ correlation and covariance are ***identical***.
- **If** X=Y then the **covariance is equivalent to variance**.
- **If** X=Y then the correlation is **auto-correlation** otherwise **cross-correlation**
- Two random variables are **uncorrelated** **if** $Corr(X,Y) = E[X.Y] = E[X].E[Y]$

Statistical: Auto Correlation ACF

- ***Autocorrelation*** is the linear dependence of a variable with a delayed copy of itself as a function of delay.
- It is a mathematical tool for finding repeating patterns, such as the presence of aperiodic signal which has been buried under noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies.
- It is often used in signal processing for analyzing functions.

Statistical: Auto Correlation ACF

- ***Autocorrelation*** for wide-sense stationary process is defined as

$$R_{XX}(\tau) = E[X(t).X(t+\tau)]$$

- For processes that are also **ergodic**, the expectation can be replaced by the limit of a time average:

$$R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)X(t+\tau)\mathrm{d}t$$ for continuos (power signal)

$$R_{YY}(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} Y(i)Y(i+k)$$ for discrete (power signal)

Statistical: Correlation

- For power signals autocorrelation at a shift $\tau = 0$, the autocorrelation is the average signal power of the signal: $R_{XX}(0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |X(t)^2| \mathrm{d}t$. Prove!

- The **cross-correlation** function between two variables X and Y

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)Y(t+\tau)\mathrm{d}t$$ for continuos (power signal)

$$R_{XY}(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X(i)Y(i+k)$$ for discrete (power signal)

Statistical: Auto Correlation ACF

- ***Autocorrelation*** for **energy** signals

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} X(t)X(t+\tau)\mathrm{d}t \quad \text{for continuos}$$

$$R_{YY}(k) = \sum_{i=-\infty}^{\infty} Y(i)Y(i+k) \quad \text{for discrete}$$

- ***Cross-correlation*** for **energy** signals

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} X(t)Y(t+\tau)\mathrm{d}t \quad \text{for continuos}$$

$$R_{XY}(k) = \sum_{i=-\infty}^{\infty} X(i)Y(i+k) \quad \text{for discrete}$$

- <u>For energy signal, autocorrelation at zero lag</u> is the energy of the signal: $R_{XX}(0) = E_X$ Prove!

Statistical: Correlation Function

Estimation of correlation functions for a finite (unlimited) length of data T, N:

$$corr(X,Y) = R_{XY}(\tau) \approx \frac{1}{T}\int_0^T X(t)Y(t+\tau)\mathrm{d}t$$ (for continuos)

$$R_{XY}(k) \approx \frac{1}{N}\sum_{i=1}^{N} X(i)Y(i+k)$$ (for discrete)

Recalls: stochastic system

- A stochastic process **X(t)** is **Wide-sense Stationary process (WSS) if** Its mean function and its correlation function do not change by time shifts:

  1. $E[X] = \mu_X(t) = \mu_X,$ for all $t \in \mathbb{R}$.
  2. $R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1),$ for all $t_1, t_2 \in \mathbb{R}$.

Properties of correlation functions

These properties hold for WSS process:

- A fundamental property of the **autocorrelation** is symmetry or it is an even function:

$$R_{XX}(\tau) = R_{XX}(-\tau) \quad \text{for all } \tau \in \mathbb{R}$$

- For the **cross-correlation:** $R_{XY}(\tau) = R_{YX}(-\tau) \quad$ for all $\tau \in \mathbb{R}$

- The continuous **autocorrelation** function reaches its peak at the origin for any delay τ

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

- For **Cross-correlation** $RXY(0)$ is not necessary $> RXY(\tau)$ for $\tau \neq 0$.

- The **autocorrelation** of a continuous-time white noise signal will have a strong peak at $\tau = 0$ and will be absolutely 0 for all other $\tau \neq 0$.

Example 1

Example: Determine the cross correlation sequence $r_{xy}(l)$

$x(n) = \{\ldots 0, 0, 2, -1, 3, 7, \underset{\uparrow}{1}, 2, -3, 0, 0 \ldots\ldots\}$

$y(n) = \{\ldots 0, 0, 1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5, 0, 0 \ldots\ldots\}$

Solution:

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

$$r_{xy}(0) = 2+1+6-14+4+2+6 = 7$$

$$r_{xy}(1) = \sum_{n=-\infty}^{\infty} x(n)y(n-1)$$

$y(n-1) = \{\ldots\ldots 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0 \ldots\ldots\ldots\}$

$x(n) = \{\ldots 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0 \ldots\ldots\}$

$$r_{xy}(1) = -1-3+14-2+8-3 = 13$$

Example 1 contd.

$r_{xy}(2) = -18 \quad r_{xy}(3) = 16 \quad r_{xy}(4) = -7 \quad r_{xy}(5) = 5 \quad r_{xy}(6) = -3 \quad r_{xy}(l) = 0 \quad l \geq 7$

$r_{xy}(-1) = 0 \quad r_{xy}(-2) = 33 \quad r_{xy}(-3) = -14 \quad r_{xy}(-4) = 36 \quad r_{xy}(-5) = 19$

$r_{xy}(-6) = -9 \quad r_{xy}(-7) = 10 \quad r_{xy}(l) = 0 \quad \text{for } l \leq -8$

$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, 0, \underset{\uparrow}{7}, 13, -18, 16, -7, 5, -3\}$

Example 2

Example: Pseudo- Random Noise

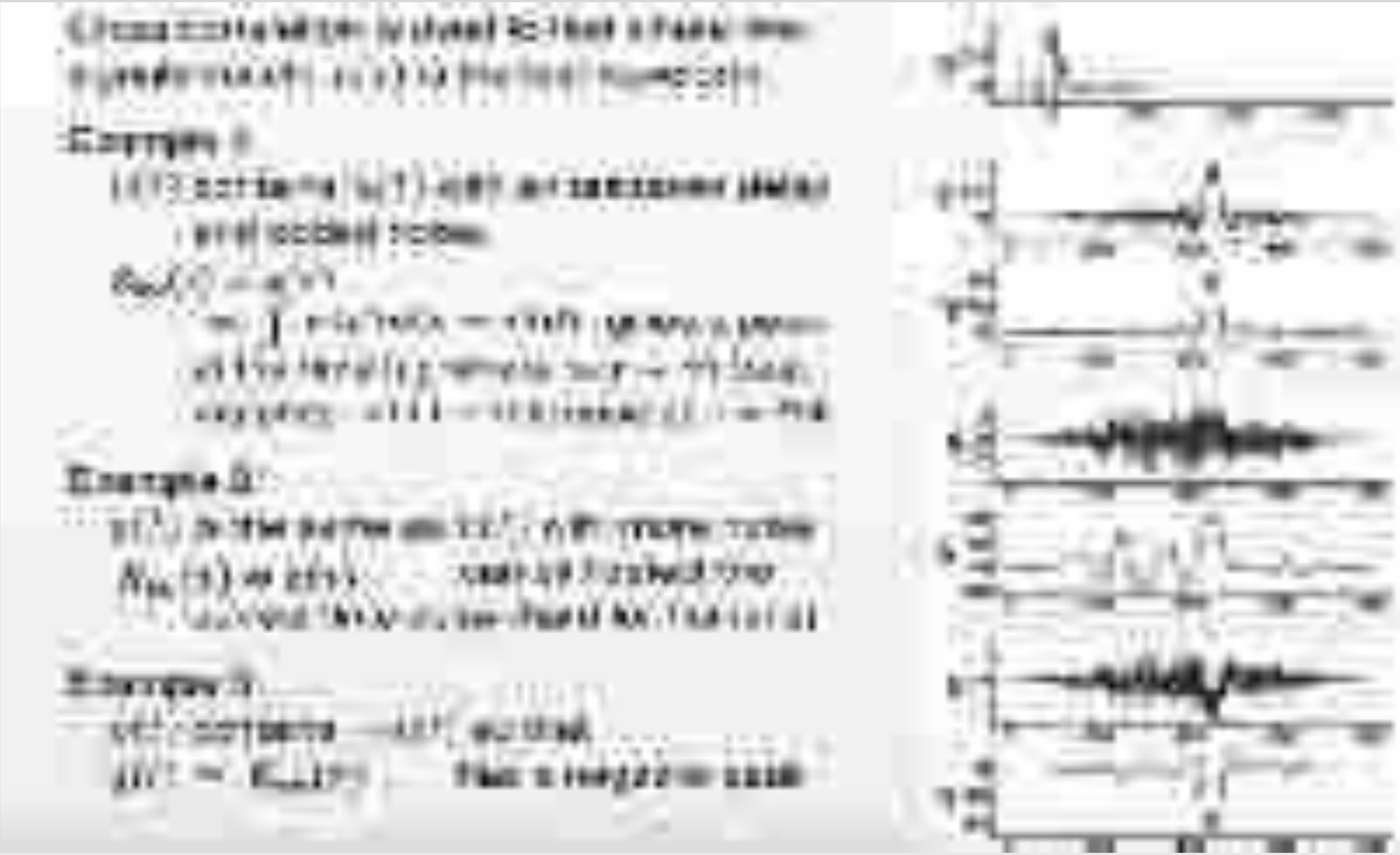
$X(n) = \{1, 1, 1, -1, -1, 1, -1\}$, find $R_{xx}(l)$, $l=0,\dots,6$

L = 0	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(0)=7$
L = 1	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(1)=0$
L = 2	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(2)= -1$
L = 3	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(3)=0$
L = 4	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(4)=-1$

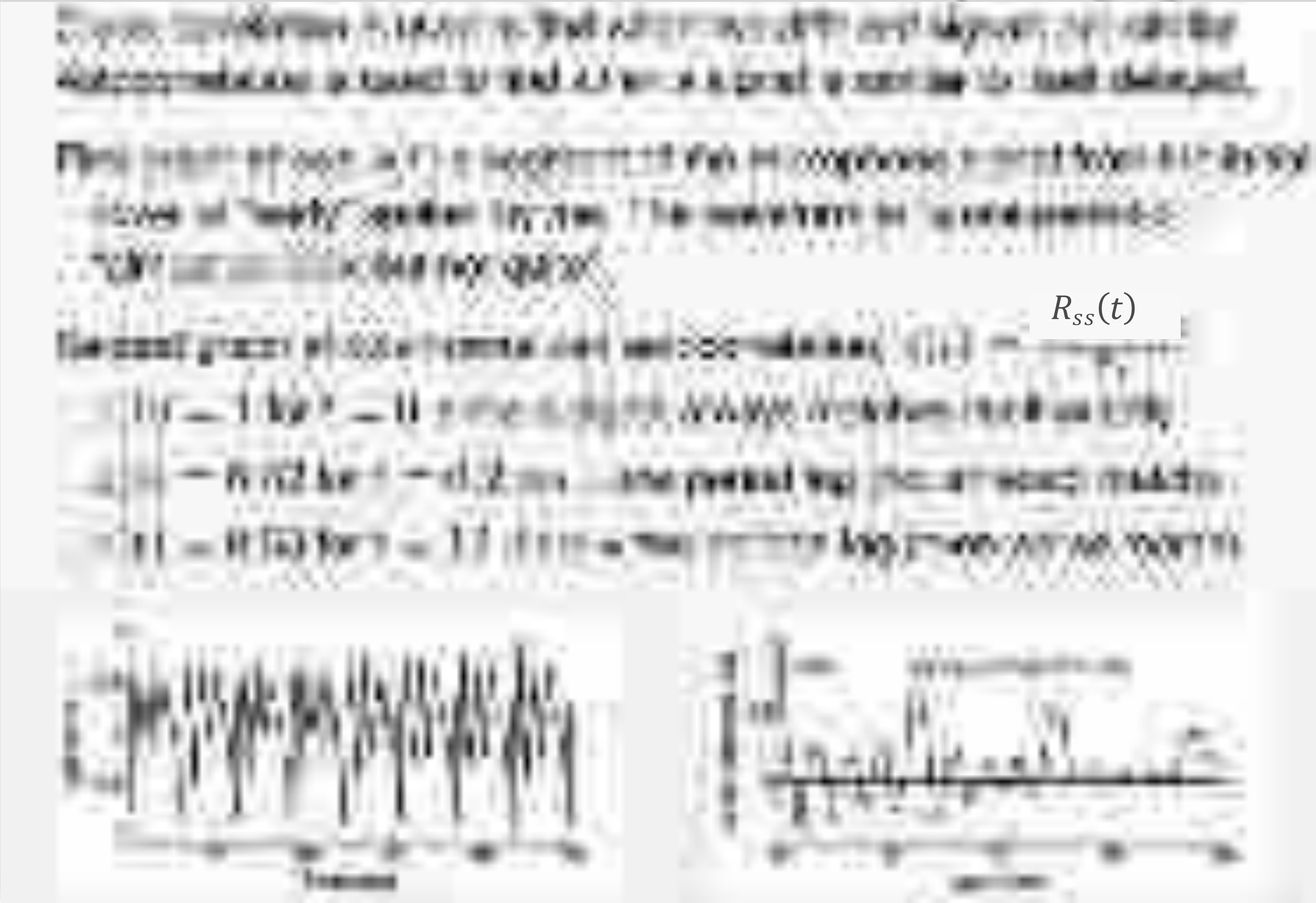
$R_{xx}(5)=0$, $R_{xx}(6)= -1$

Examples 1 & 2 are taken from lecture notes of Auto and cross correlation by A. Abu-Hudrouss, Islamic University Gaza, 2008.

Cross-correlation: Signal Matching [3]



Autocorrelation example [3]



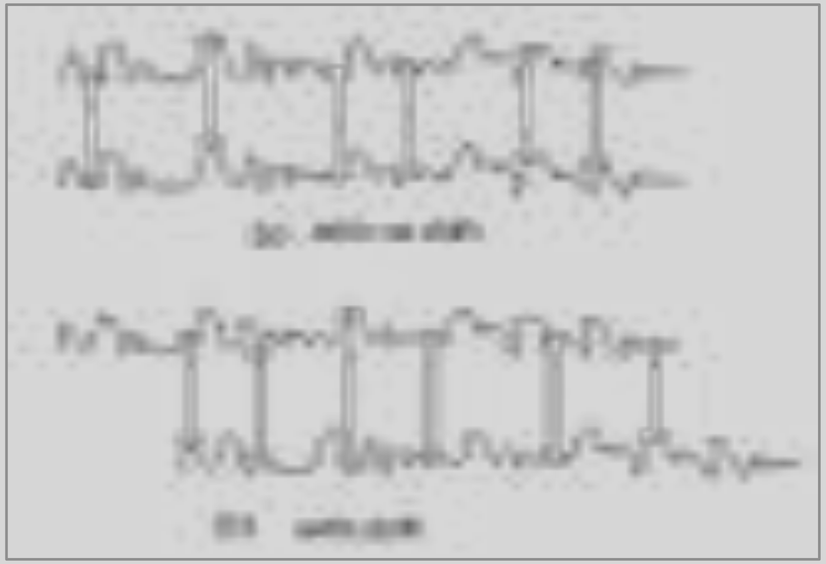
Statistical: ACF Application

White Noise

- The term "**white**" refers to the frequency domain characteristic of noise.
- The term ***white noise*** is analogous to *white light* which contains all visible light frequencies.
- Ideal white noise has equal power per unit bandwidth, which results in a **flat power spectral density (PSD) across the frequency range of interest**.
- PSD shows the strength of the variations (strong/weak energy) as a function of frequency.
- For ***white noise***, the power in the frequency range from 100 Hz to 110 Hz **is the same** as the power in the frequency range from 1000 Hz to 1010 Hz.

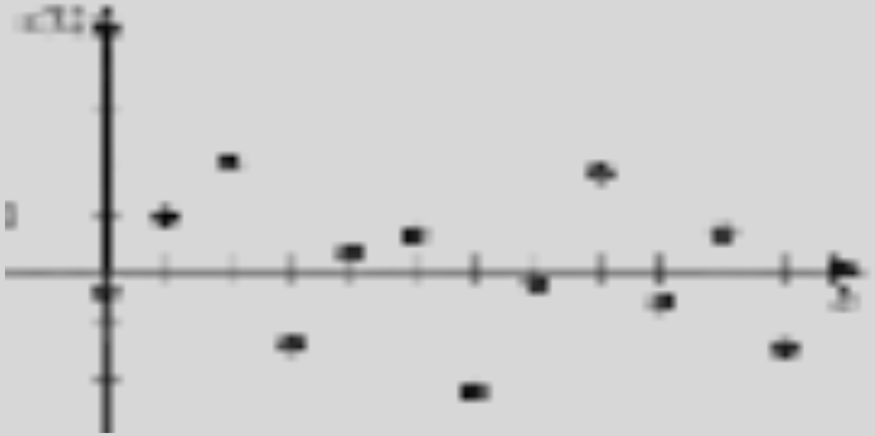
White Noise

- Ideal white noise has zero mean: $E[X(t)] = 0$.
- A signal $X(t)$ is called a ***white noise sequence*** if **X** is a sequence <u>of independent random variables</u>, i.e. $E[(X(t)X(t-\tau))] = Rxx(\tau)=0$ for all $\tau \neq 0$.



If the noise has a mean of zero, there will be as many positive products as there are negative products. Hence they will all sum up to zero.

White Noise

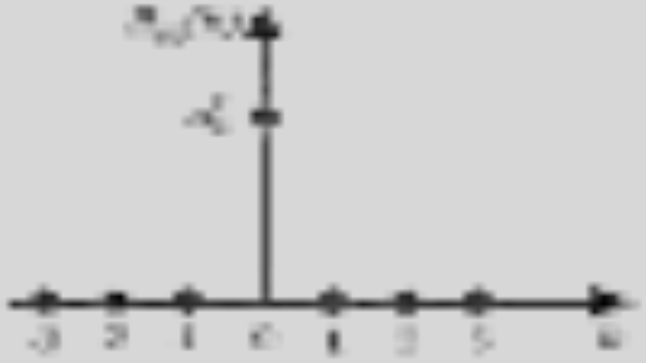


Example : zero-mean discrete white noise

$$\mathbb{E}\{e(k)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e(k) = 0$$

$$R_{ee}(h) = \mathbb{E}\{e(k)e(k+h)\} = 0 \quad \text{for } h \neq 0$$

$$\mathbb{E}\{e^2(k)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^2(k) = \sigma_e^2$$



Autocorrelation

Statistical: The Standard Deviation

- The most common way to describe the range of variation *is standard deviation* (**σ**).
- The standard deviation is simply the ***positive square root*** of ***the variance***.

…____________________…

Statistical: The Standard Deviation

- Example [1]: Suppose a train arrives shortly after 1:00 PM each day, and that the number of minutes after 1:00 that the train arrives can be modeled as a continuous random variable with density

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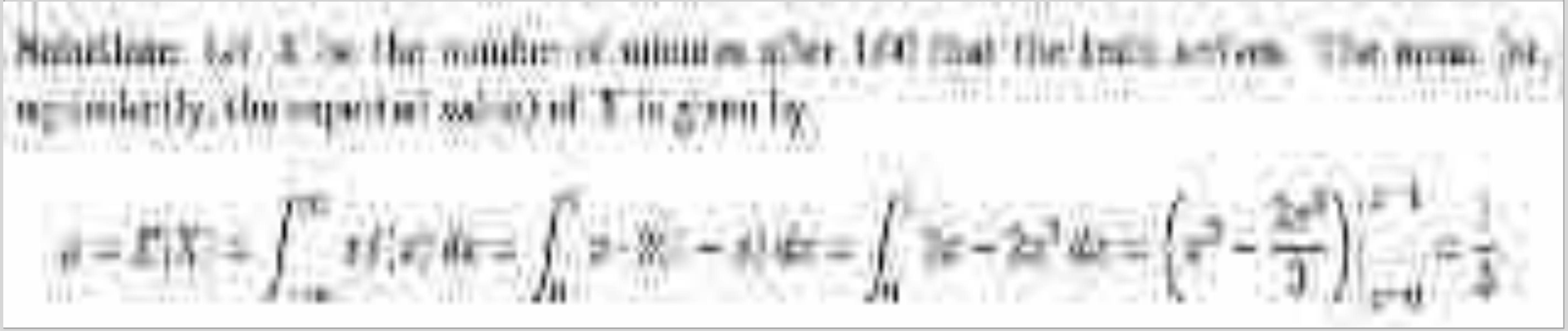
Find the mean and standard deviation of the number of minutes after 1:00 that the train arrives.



Statistical: The Standard Deviation

- Example [1]: cntd..

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Statistical: Correlation & Covariance [2]

$$Corr(X,Y) = E[X.Y]$$

$$Cov(X,Y) = E[(X - E[X]).(Y - E[Y])]$$

- **If** $\mathrm{E}[X] = E[Y] = 0,$ correlation and covariance are ***identical***.
- **If** X=Y then the **covariance is equivalent to variance**.
- **If** X=Y then the correlation is **auto-correlation** otherwise **cross-correlation**
- Two random variables are **uncorrelated** **if** $Corr(X,Y) = E[X.Y] = E[X].E[Y]$

Statistical: Auto Correlation ACF

- ***Autocorrelation*** is the linear dependence of a variable with a delayed copy of itself as a function of delay.
- It is a mathematical tool for finding repeating patterns, such as the presence of aperiodic signal which has been buried under noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies.
- It is often used in signal processing for analyzing functions.

Statistical: Auto Correlation ACF

- ***Autocorrelation*** for wide-sense stationary process is defined as

$$R_{XX}(\tau) = E[X(t).X(t+\tau)]$$

- For processes that are also **ergodic**, the expectation can be replaced by the limit of a time average:

$$R_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)X(t+\tau)\mathrm{d}t$$ for continuos (power signal)

$$R_{YY}(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} Y(i)Y(i+k)$$ for discrete (power signal)

Statistical: Correlation

- For power signals autocorrelation at a shift $\tau = 0$, the autocorrelation is the average signal power of the signal: $R_{XX}(0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |X(t)^2| \mathrm{d}t$. Prove!

- The **cross-correlation** function between two variables X and Y

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)Y(t+\tau)\mathrm{d}t$$ for continuos (power signal)

$$R_{XY}(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X(i)Y(i+k)$$ for discrete (power signal)

Statistical: Auto Correlation ACF

- ***Autocorrelation*** for **energy** signals

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} X(t)X(t+\tau)\mathrm{d}t \qquad \text{for continuos}$$

$$R_{YY}(k) = \sum_{i=-\infty}^{\infty} Y(i)Y(i+k) \qquad \text{for discrete}$$

- ***Cross-correlation*** for **energy** signals

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} X(t)Y(t+\tau)\mathrm{d}t \qquad \text{for continuos}$$

$$R_{XY}(k) = \sum_{i=-\infty}^{\infty} X(i)Y(i+k) \qquad \text{for discrete}$$

- For energy signal, autocorrelation at zero lag is the energy of the signal: $R_{XX}(0) = E_X$ Prove!

Statistical: Correlation Function

Estimation of correlation functions for a finite (unlimited) length of data T, N:

$$corr(X,Y) = R_{XY}(\tau) \approx \frac{1}{T}\int_0^T X(t)Y(t+\tau)\mathrm{d}t$$ (for continuos)

$$R_{XY}(k) \approx \frac{1}{N}\sum_{i=1}^{N} X(i)Y(i+k)$$ (for discrete)

Recalls: stochastic system

- A stochastic process **X(t)** is **Wide-sense Stationary process (WSS) if** Its mean function and its correlation function do not change by time shifts:

  1. $E[X] = \mu_X(t) = \mu_X,$ for all $t \in \mathbb{R}$.
  2. $R_{XX}(t_1, t_2) = R_{XX}(t_2 - t_1),$ for all $t_1, t_2 \in \mathbb{R}$.

Properties of correlation functions

<u>These properties hold for WSS process:</u>

- A fundamental property of the **autocorrelation** is symmetry or it is <u>an even function</u>:

$$R_{XX}(\tau) = R_{XX}(-\tau) \quad \text{for all } \tau \in \mathbb{R}$$

- For the **cross-correlation:** $R_{XY}(\tau) = R_{YX}(-\tau)$ for all $\tau \in \mathbb{R}$

- The continuous **autocorrelation** function reaches its peak at the origin for any delay τ

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

- For **Cross-correlation** $RXY(0)$ is not necessary $> RXY(\tau)$ for $\tau \neq 0$.

- The **autocorrelation** of a continuous-time <u>white noise</u> signal will have a strong peak at $\tau = 0$ and will be absolutely 0 for all other $\tau \neq 0$.

Example 1

Example: Determine the cross correlation sequence $r_{xy}(l)$

$x(n) = \{\ldots 0, 0, 2, -1, 3, 7, \underset{\uparrow}{1}, 2, -3, 0, 0, \ldots\}$

$y(n) = \{\ldots 0, 0, 1, -1, 2, -2, \underset{\uparrow}{4}, 1, -2, 5, 0, 0, \ldots\}$

Solution:

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

$$r_{xy}(0) = 2+1+6-14+4+2+6 = 7$$

$$r_{xy}(1) = \sum_{n=-\infty}^{\infty} x(n)y(n-1)$$

$y(n-1) = \{\ldots 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0 \ldots\}$

$x(n) = \{\ldots 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0 \ldots\}$

$$r_{xy}(1) = -1-3+14-2+8-3 = 13$$

Example 1 contd.

$r_{xy}(2) = -18 \quad r_{xy}(3) = 16 \quad r_{xy}(4) = -7 \quad r_{xy}(5) = 5 \quad r_{xy}(6) = -3 \quad r_{xy}(l) = 0 \quad l \geq 7$

$r_{xy}(-1) = 0 \quad r_{xy}(-2) = 33 \quad r_{xy}(-3) = -14 \quad r_{xy}(-4) = 36 \quad r_{xy}(-5) = 19$

$r_{xy}(-6) = -9 \quad r_{xy}(-7) = 10 \quad r_{xy}(l) = 0 \quad for\ l \leq -8$

$r_{xy}(l) = \{10, -9, 19, 36, -14, 33, 0, 7, 13, -18, 16, -7, 5, -3\}$

$\uparrow$ (at $r_{xy}(0) = 7$)

Example 2

Example: Pseudo- Random Noise

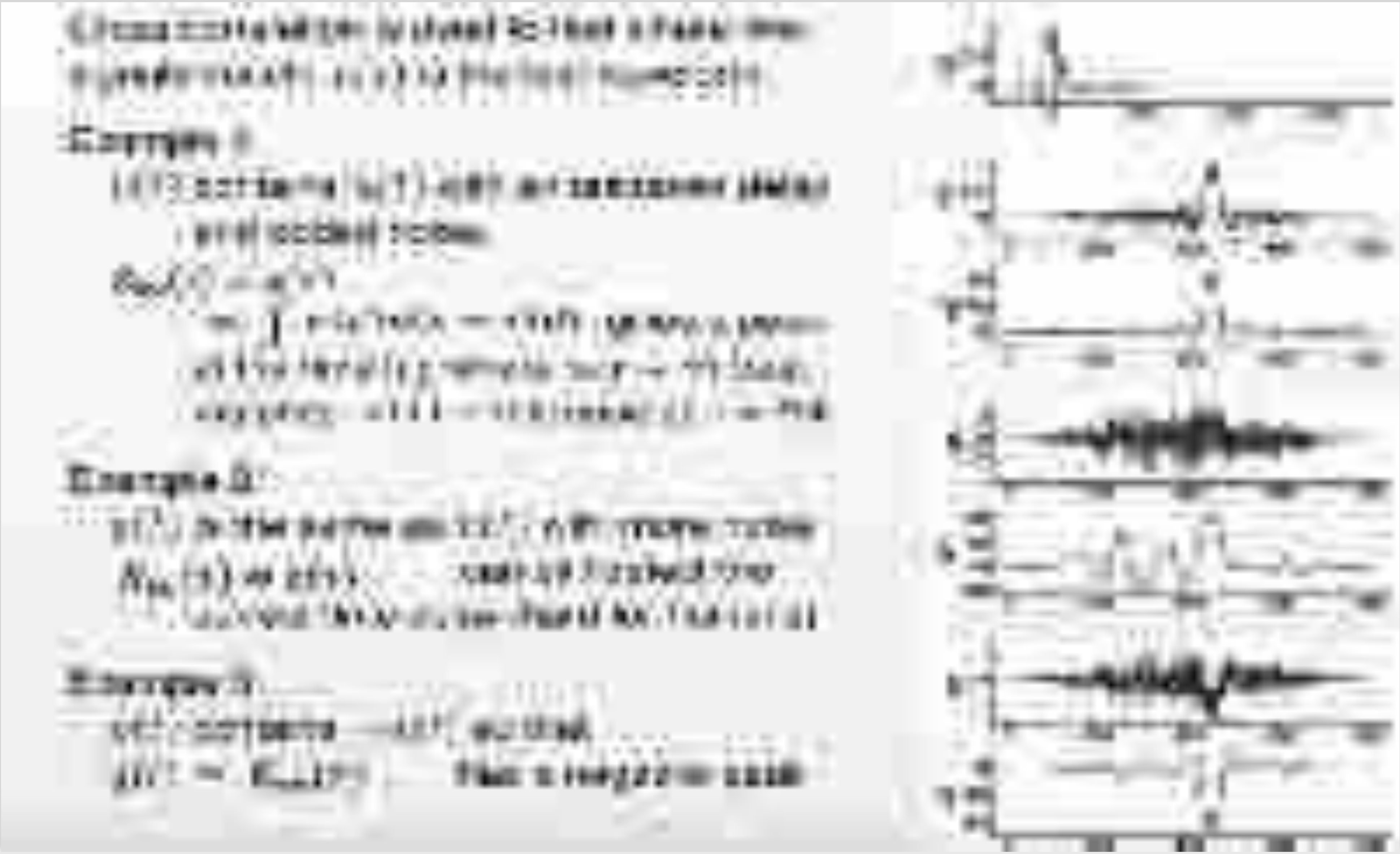
$X(n) = \{1, 1, 1, -1, -1, 1, -1\}$, find $R_{xx}(l)$, $l=0,\ldots,6$

L = 0	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(0)=7$
L = 1	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(1)=0$
L = 2	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(2)=-1$
L = 3	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(3)=0$
L = 4	{+1, +1, +1, -1, -1, +1, -1}	$R_{xx}(4)=-1$

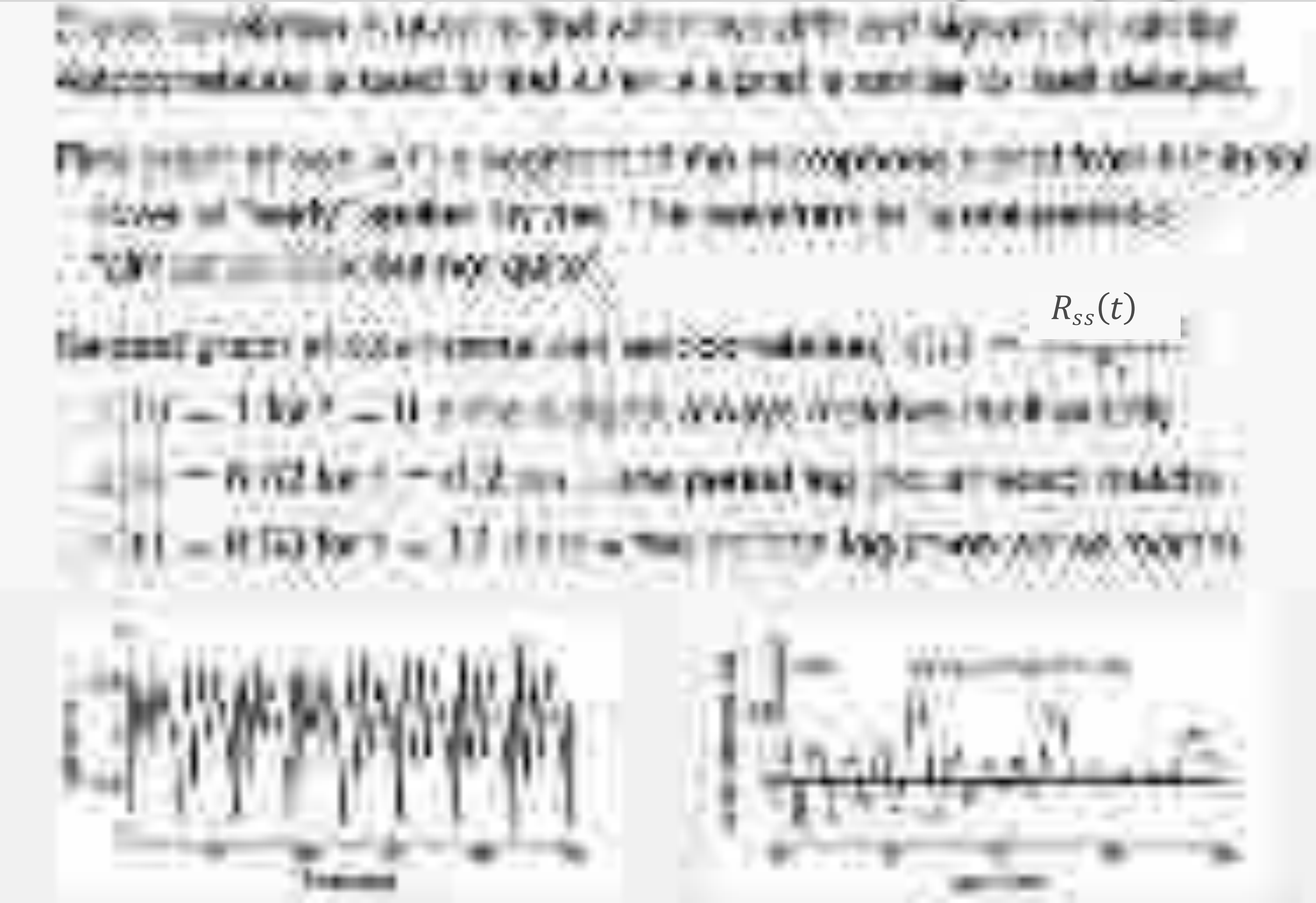
$R_{xx}(5)=0$, $R_{xx}(6)=-1$

Examples 1 & 2 are taken from lecture notes of Auto and cross correlation by A. Abu-Hudrouss, Islamic University Gaza, 2008.

Cross-correlation: Signal Matching [3]



Autocorrelation example [3]



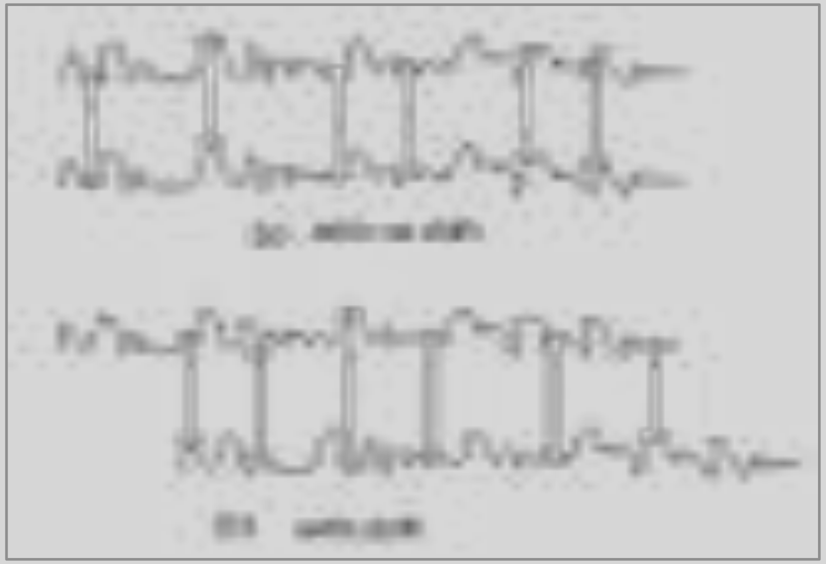
Statistical: ACF Application

White Noise

- The term "**white**" refers to the frequency domain characteristic of noise.
- The term ***white noise*** is analogous to *white light* which contains all visible light frequencies.
- Ideal white noise has equal power per unit bandwidth, which results in a **flat power spectral density (PSD) across the frequency range of interest**.
- PSD shows the strength of the variations (strong/weak energy) as a function of frequency.
- For ***white noise***, the power in the frequency range from 100 Hz to 110 Hz **is the same** as the power in the frequency range from 1000 Hz to 1010 Hz.

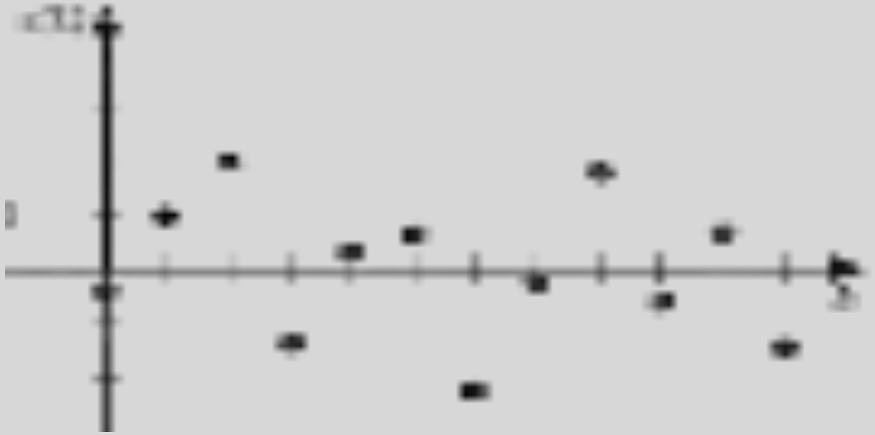
White Noise

- Ideal white noise has zero mean: $E[X(t)] = 0.$
- A signal $X(t)$ is called a ***white noise sequence*** if **X** is a sequence of independent random variables, i.e. $E[(X(t)X(t-\tau))] = \text{Rxx}(\tau)=0$ for all $\tau \neq 0.$



If the noise has a mean of zero, there will be as many positive products as there are negative products. Hence they will all sum up to zero.

White Noise

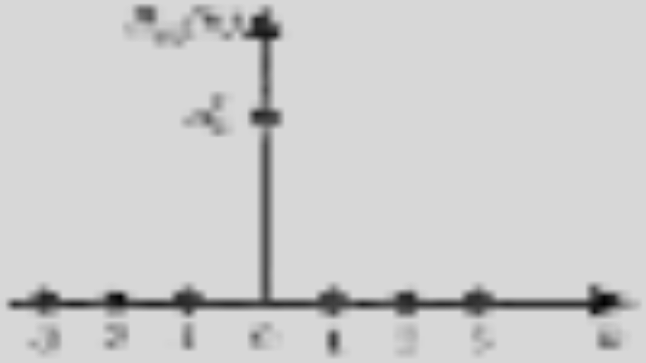


Example : zero-mean discrete white noise

$$\mathbb{E}\{e(k)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e(k) = 0$$

$$R_{ee}(h) = \mathbb{E}\{e(k)e(k+h)\} = 0 \quad \text{for } h \neq 0$$

$$\mathbb{E}\{e^2(k)\} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^2(k) = \sigma_e^2$$



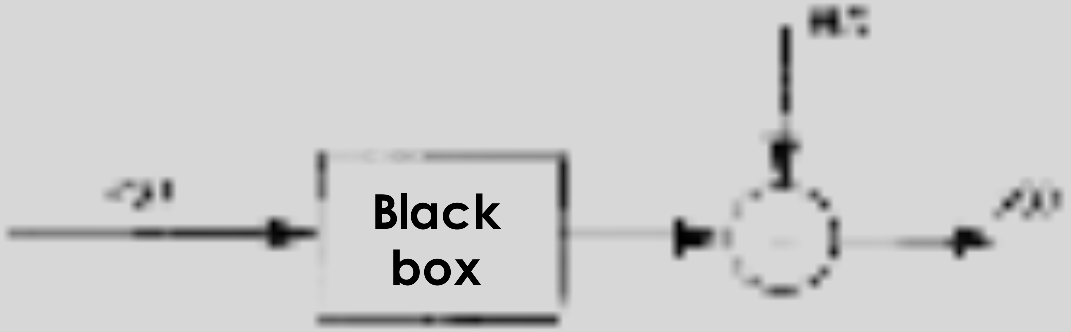
Autocorrelation

System Identification

A Third-year Course for Control and Mechatronics Engineering

By Dr. Taghreed M. MohammadRidha

Problem Formulation



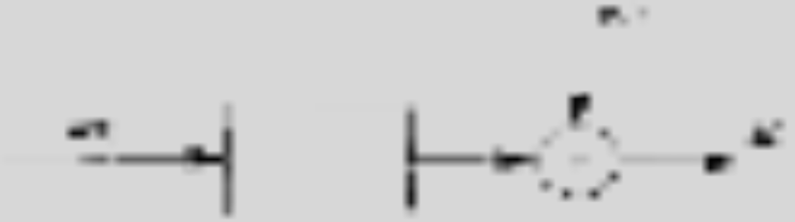
- Actual system is Linear Time Invariant LTI.
- Process $y(t) = g(t) \otimes u(t) + v(t)$, where $\otimes$ is the convolution operator.
- Estimates of $g(t)$ using time domain nonparametric methods.
- Test the error $|g(t) - \hat{g}(t)|$ for all $t \geq 0$.

Transient Analysis

- In transient analysis, we determine the system’s response to a particular signal (typically a pulse or a step signal).
- By applying a pulse or step input signal to the system, the pulse or step response can be observed from the output.
- Generally this provides good insights into important properties of the system, as e.g. the presence and length of time delays, static gain and time constants.

Transient Analysis [4,5]

- **Impulse Response (discrete)**

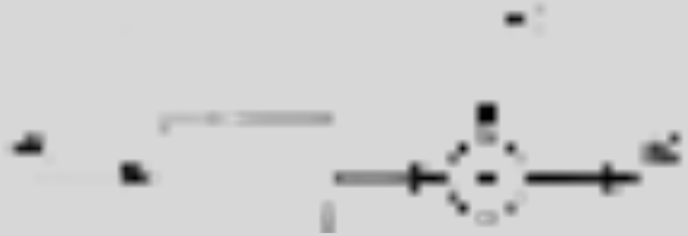


$$y(kT) = \sum_{i=0}^{\infty} g(i)u(kT - i) + v(kT)$$

- $g(k) = \sum_{i=1}^{\infty} g(i)$ is the impulse response, $k = 0, 1, 2, \ldots$ is the sampling instants, T is sampling time.
- For ease of notation, assume T is one time unit and use k to enumerate sampling instants.
- For identification, the input u and output y are recorded and the system g(k) will be modelled with a ***Finite Impulse Response* (FIR)** $\hat{g}(k)$, ***k = 0, 1, 2, ...,M***.
- For a causal system the lower limit of the summation can not be less than zero.

Transient Analysis [4]

- **Impulse Response Analysis (discrete)**



$$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i) + v(k)$$

If the above system is subjected to a pulse I/P: $\mathrm{u(k)} = \begin{cases} \alpha, & k = 0 \\ 0, & k \neq 0 \end{cases}$

Then $$y(k) = \alpha g(k) + v(k)$$ **Prove!**

If the signal to noise ratio (SNR) is High, then the impulse response can be estimated:

$$\hat{g}(k) = \frac{y(k)}{\alpha}$$

Transient Analysis [4]

- **Impulse Response Analysis (discrete)**

If the signal to noise ratio (SNR) is High, then the impulse response can be estimated:

$$\hat{g}(k) = \frac{y(k)}{\alpha}$$

With error:

$$|g(k) - \hat{g}(k)| = \frac{|v(\mathrm{k})|}{\alpha}$$

➢ **Weakness**

➢ For small error, **α** must be very HIGH.

➢ Many physical process do not allow such high pulse.

➢ Such an input may cause unwanted nonlinear dynamic behavior that would disturb the linearized behavior already set to the model.

Transient Analysis [4]

- **Step Response Analysis (discrete)**

$$y(k) = \sum_{i=0}^{\infty} g(i)u(k-i) + v(k) \qquad (1)$$

If the system in eq. (1) is subjected to a **step** I/P: $u(k) = \begin{cases} \alpha, & k \geq 0 \\ 0, & k < 0 \end{cases}$

Then $$y(k) = \alpha \sum_{i=0}^{k} g(i) + v(k)$$

Prove!

From this, estimates of $g(k)$ can be obtained as:

$$\hat{g}(k) = \frac{y(k) - y(k-1)}{\alpha}$$

Prove!

With error $$|g(k) - \hat{g}(k)| = \frac{|v(k) - v(k-1)|}{\alpha}$$

Transient Analysis [4]

- **Step Response Analysis (discrete)**

✓ Practical for observing general features (time delay, static gain, response shape..).

Weakness

➢ Estimation of impulse response coefficients suffer from large error term.

Correlation Analysis [4]

- **Correlation Analysis (Discrete)**

$$\boldsymbol{y}(\boldsymbol{k}) = \boldsymbol{g}(\boldsymbol{k}) \otimes \boldsymbol{u}(\boldsymbol{k}) + \boldsymbol{v}(\boldsymbol{k})$$

Let $\boldsymbol{u}(\boldsymbol{k})$ be a random Wide Sense Stationary (WSS) process **independent** of noise $\boldsymbol{v}(\boldsymbol{k})$

then $\boldsymbol{y}(\boldsymbol{k})$, is also a random process.

Multiply $\boldsymbol{y}(\boldsymbol{k})$ by $u(\boldsymbol{k}+\boldsymbol{\tau})$ and take the expected value :

$$E[y(k)u(k+\tau)] = \sum_{i=0}^{\infty} g(i)\, E[u(k-i)u(k+\tau)] + E[v(k)u(k+\tau)]$$

Correlation Analysis [4]

$$E[y(k)u(k+\tau)] = \sum_{i=0}^{\infty} g(i)\, E[u(k-i)u(k+\tau)] + E[v(k)u(k+\tau)]$$

As $u(t)$ *and* $v(t)$ are completly uncorrelated then $E[v(k)u(k+\tau)] \equiv 0$ (or $R_{vu}(\tau) = 0$) thus

$$R_{yu}(\tau) = \sum_{i=0}^{\infty} g(i) R_{uu}(i+\tau)$$

From correlation properties, $R_{yu}(-\tau) = R_{uy}(\tau)$ then

$$R_{yu(-\tau)} = R_{uy}(\tau) = \sum_{i=0}^{\infty} g(i) R_{uu}(i-\tau)$$

Again using the property $R_{uu}(-\tau) = R_{uu}(\tau)$, then $\boldsymbol{R_{uu}(i-\tau) = R_{uu}(\tau-i)}$ then

$$R_{uy}(\tau) = \sum_{i=0}^{\infty} g(i) R_{uu}(\tau-i)$$

Or $$\boldsymbol{R_{uy}(\tau) = g(\tau) \otimes R_{uu}(\tau)} \quad (2)$$

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Correlation Method [4]

$$\boldsymbol{R_{uy}(\tau) = g(\tau) \otimes R_{uu}(\tau)} \qquad (2)$$

To solve g(k) from eq. (2) two cases will be distinguished:

1. If u(k) is a <u>white noise</u> sequence with $\mathrm{E}[u^2(k)] = \sigma^2$:

$$R_{uy}(\tau) = \hat{\boldsymbol{g}}(\tau) \otimes R_{uu}(0) = \hat{\boldsymbol{g}}(\tau)\sigma^2$$

$$\hat{\boldsymbol{g}}(\boldsymbol{\tau}) = \frac{\boldsymbol{R_{uy}(\tau)}}{\boldsymbol{\sigma^2}}$$

With the assumption the process is ergodic, then from ***N measurements*** the estimated crosscorrelation is

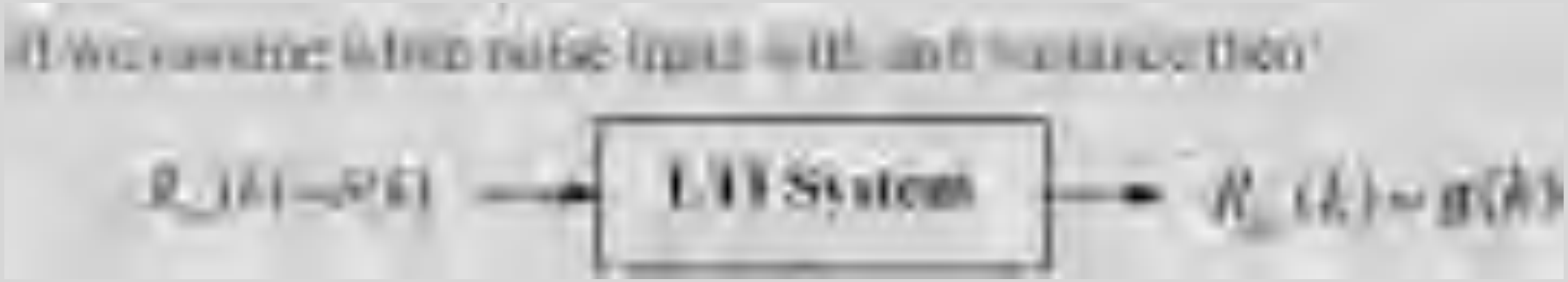
$$\boldsymbol{R_{uy}(\tau) = E[u(k)y(k+\tau)] = \frac{1}{N}\sum_{i=0}^{N-1} u(i)y(i+\tau)}$$

Correlation Method [4]

1. If u(k) is a <u>white noise</u> sequence with $\mathrm{E}[u^2(k)] = \sigma^2$:

$$\hat{g}(\tau) = \frac{R_{uy}(\tau)}{\sigma^2}$$

Where $$R_{uy}(\tau) = \frac{1}{N}\sum_{i=0}^{N-1} u(i)y(i+\tau)$$



$$\boxed{\hat{g}(\tau) = \frac{1}{N}\sum_{i=0}^{N-1} u(i)y(i+\tau)}$$

Correlation Method [4]

$$R_{uy}(\tau) = g(\tau) \otimes R_{uu}(\tau)$$

2. *If u(k) is not a <u>white noise</u> signal:*

- <u>Solution 1:</u>

  i) Estimate the correlation function

$$R_{uu}(\tau) = \frac{1}{N}\sum_{i=0}^{N-1} u(i)u(i+\tau)$$

ii) and next **solve** the linear set of M equations for $\hat{g}(k)$:

$$R_{uy}(\tau) = \sum_{i=0}^{M-1} \hat{g}(i)R_{uu}(\tau - i) \quad or \quad R_{uy}(\tau) = \hat{g}(\tau) \otimes R_{uu}(\tau)$$

Correlation Method [5]

2. *How to estimate $\hat{g}(k)$ If u(k) is not a <u>white noise</u> signal* (Solution1):

$$R_{uy}(\tau) = \sum_{i=0}^{M-1} \hat{g}(i) R_{uu}(\tau - i)$$

In matrix form

$$\begin{pmatrix} R_{uy}(0) \\ \vdots \\ R_{uy}(M-1) \end{pmatrix} = \begin{pmatrix} R_u(0) & R_u(-1) & R_u(-(M-1)) \\ R_u(1) & R_u(0) & R_u(-(M-2)) \\ \vdots & & \vdots \\ R_u(M-1) & \dots & R_u(0) \end{pmatrix} \begin{pmatrix} \hat{g}(0) \\ \vdots \\ \hat{g}(M-1) \end{pmatrix}$$

where $R_u(\tau) = R_u(-\tau)$.

Correlation Method [5]

2. *If u(k) is not a <u>white noise</u> signal* (Solution1):

$$\begin{pmatrix} R_{uy}(0) \\ \vdots \\ R_{uy}(M-1) \end{pmatrix} = \begin{pmatrix} R_{uu}(0) & R_{uu}(-1) & R_{uu}(-(M-1)) \\ R_{uu}(1) & R_{uu}(0) & R_{uu}(-(M-2)) \\ \vdots & & \vdots \\ R_{uu}(M-1) & \cdots & R_{uu}(0) \end{pmatrix} \begin{pmatrix} g(0) \\ \vdots \\ g(M-1) \end{pmatrix}$$

$$\text{where } R_{uu}(\tau) = R_{uu}(-\tau).$$

a) First find $\boldsymbol{R_{uy}(\tau), R_{uu}(\tau)}$ where $\boldsymbol{\tau = 0, \ldots, M-1}$.

b) Next, solve the linear set of **M** equations for $\boldsymbol{\hat{g}(\tau)}$.

<u>NOTE</u>

For stable systems (without integrator), impulse response goes asymptotically to zero such that we can suppose that M is sufficiently large $\boldsymbol{g(k) = 0}$ for k>M

Correlation Method Analysis [6]

2. *If u(k) is not a white noise signal*

- <u>Solution 2</u>: ***Filter input and output with a pre-whitening filter:***

Suppose we know a filter $L(z)$, such that $u_F(k) = L(z)u(k)$ is a white noise signal.

The filtered output can be written as

$$y_F(t) = L(z)\left(G_T(z)u(t) + v(t)\right).$$

For a linear system then also

$$y_F(t) = G_T(z)u_F(t) + L(z)v(t).$$

Then using y_F and u_F, $\hat{\boldsymbol{g}}(\boldsymbol{k})$ can be estimated: $\hat{\boldsymbol{g}}(\boldsymbol{\tau}) = \frac{R_{u_F y_F}(\boldsymbol{\tau})}{\boldsymbol{\sigma}^2}$

Correlation Method Analysis [6]

2. *If u(k) is not a <u>white noise</u> signal-* (Solution 2)

- Finding ***a pre-whitening filter*** $L(z)$**:**

Try to use a linear model of order *n*:

$$\boldsymbol{L(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

Where $f(k)z^{-i} = f(k-i)$.

And look for a best fit of ***n*** and $\boldsymbol{a_n}$ such that the autocorrelation of input is minimized **(**such that $\boldsymbol{u_F}$ is <u>as white as possible</u>**)**.

Correlation Method Analysis [6]

Example: Using the following data, find the impulse response g(k) by Correlation method:

$u(k) = 1,0,\ldots,0$ and $y(k) = 1,\frac{1}{2},\frac{1}{4},\ldots,\frac{1}{2^{N-1}}$.

Sol. Since the input u(k) is not white noise, we can solve for g(k) by using matrix form solution:

a) First find $\boldsymbol{R_{uy}(\tau)}, \boldsymbol{R_{uu}(\tau)}$ where $\boldsymbol{\tau = 0,\ldots, M-1, M = 2}$.

$$\begin{bmatrix} R_{uy}(0) \\ R_{uy}(1) \end{bmatrix} = \begin{bmatrix} R_{uu}(0) & R_{uu}(1) \\ R_{uu}(1) & R_{uu}(0) \end{bmatrix} \cdot \begin{bmatrix} \hat{g}(0) \\ \hat{g}(1) \end{bmatrix}, \qquad \begin{bmatrix} \hat{g}(0) \\ \hat{g}(1) \end{bmatrix} = \begin{bmatrix} R_{uu}(0) & R_{uu}(1) \\ R_{uu}(1) & R_{uu}(0) \end{bmatrix}^{-1} \cdot \begin{bmatrix} R_{uy}(0) \\ R_{uy}(1) \end{bmatrix},$$

$$R_{uy}(0) = \frac{1}{2}\sum_{i=0}^{1} u(i)y(i) = \frac{1}{2}, \qquad R_{uy}(1) = \frac{1}{2}\sum_{i=0}^{1} u(i)y(i+1) = \frac{1}{4}.$$

$$R_{uu}(0) = \frac{1}{2}\sum_{i=0}^{1} u(i)^2 = \frac{1}{2}, \qquad R_{uu}(1) = \frac{1}{2}\sum_{i=0}^{1} u(i)u(i+1) = 0 \quad \Rightarrow \quad \begin{bmatrix} \hat{g}(0) \\ \hat{g}(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

Homework: find $\hat{g}(\tau)\ for\ \boldsymbol{\tau = 0,\ldots, M-1, M = 5}$. Draw $\hat{g}(\tau)$.

Observations about transient and Correlation analysis

Transient response analysis

✓ widely used, fast and simple method to gain insight in system dynamics.

Weakness

➢ hard to determine accurate model (limited input signal size, disturbances, noise).

Correlation analysis

✓ does not require special input signals (such as impulses).

✓ can compensate low SNR by longer measurement periods.

PSEUDO-RANDOM BINARY SEQUENCE

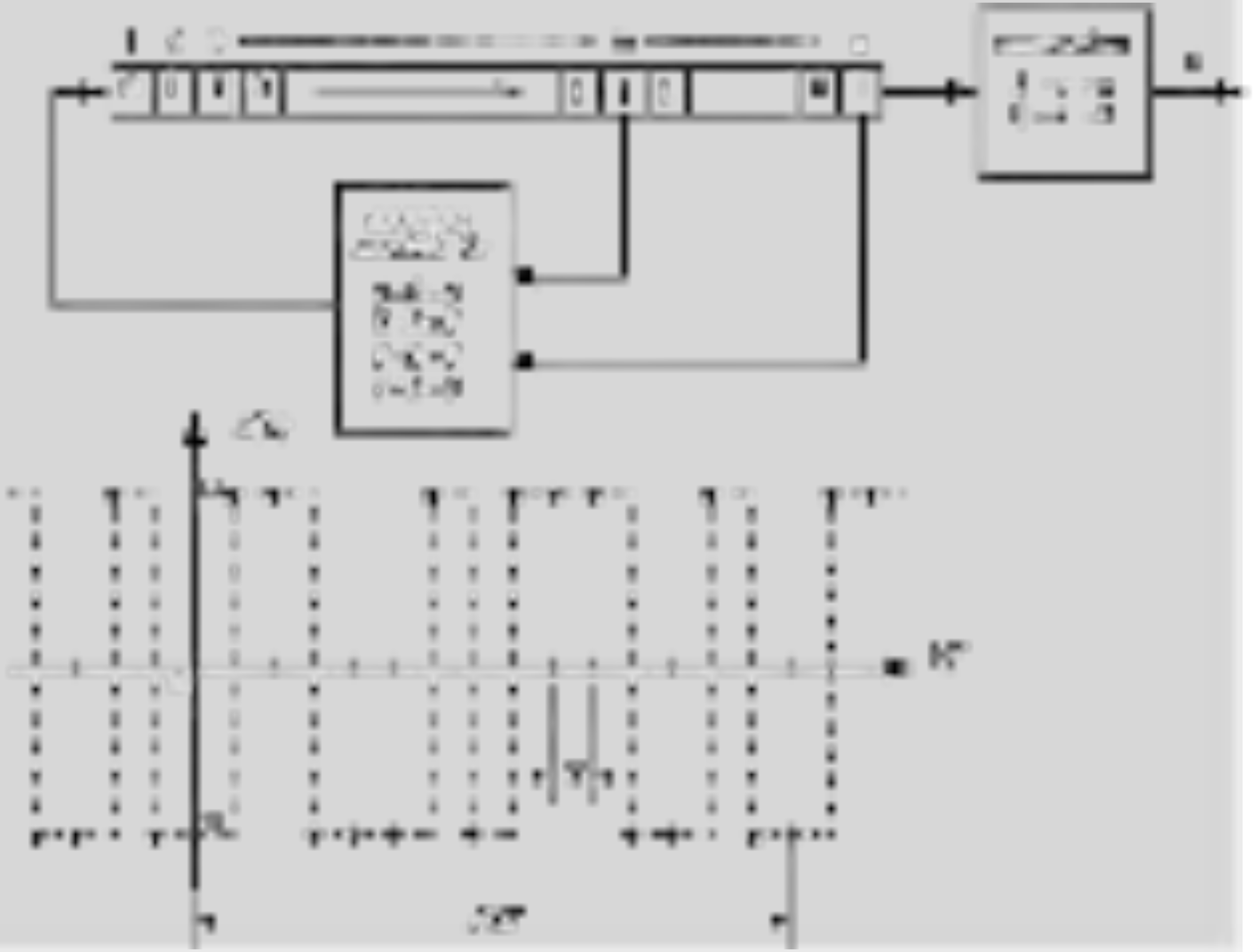
One of the useful periodic signals for system identification work is the Pseudo-random Binary Sequence (PRBS).

To produce a PRBS we need:

- **n**-bit shift registers and an XOR gate
- The max. number of states

(sequence length) for **n** shift registers is

N = 2 n -1



PSEUDO-RANDOM BINARY SEQUENCE

PRBS has the following properties:

- ✓ The PRBS characteristics are very similar to those of white noise.
- ✓ APRBS can be used as a test signal instead of white noise provided that it is chosen to have a P.D.S which is uniform over the B.W of the system.
- ✓ PRBS are more often used than white noise.

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