Course No-iii In Control Theory

Frequency Response Analysis

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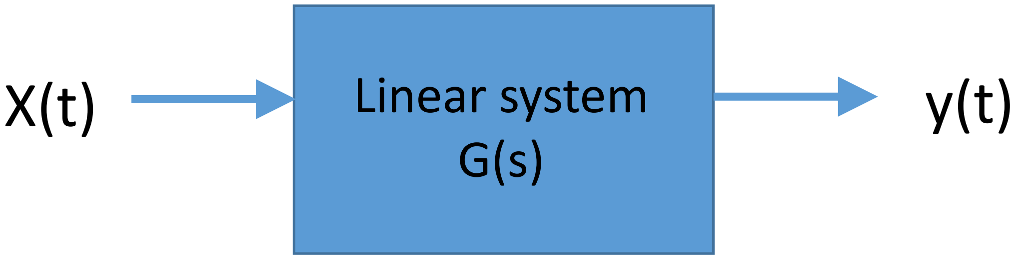
Introduction:

The magnitude and phase relationship between sinusoidal input and steady state output of a system is known as frequency response.

Note: it is independent of the amplitude and phase of the input signal



If the input signal is $x(t) = X \sin(wt)$

The output can be written as; $y(t) = Y \sin(wt + \phi)$

Where, $Y = X\,|G(jw)|$

$\emptyset = <G(jw)$

$|G(jw)|$ is the ratio of the amplitude of the output sinusoid to the input sinusoid

$\emptyset = \angle G(jw)$ is the phase shift of the output sinusoid with respect to the input sinusoid

G(jw) is called the sinusoidal transfer function and is obtained from expression

$$G(s)=\frac{Y(s)}{X(s)}$$

By putting ‘jw’ for ‘s’

$$G(jw)=\frac{Y(jw)}{X(jw)}$$ is the sinusoidal transfer function

The main features of the frequency response method

i) Sinewave is easily generated in laboratory unlike other test inputs like step or impulse.

ii) The frequency response off a system can be obtained in simple way.

iii) Transfer function of the complicated system can be determined experimentally by frequency response.

Calculation of frequency response

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ Example:

$G(s)=\frac{5}{(s+1)}$

Put s=jw

$G(jw)=\frac{5}{(jw+1)}$

$|G(jw)| = \frac{5}{\sqrt{w^2+1}}$, $\angle G(jw) = -\tan^{-1}(w)$

Example: Consider the system shown below.
The transfer function G(s) is;

$$G(s)=\frac{K}{(Ts+1)}$$

For sinusoidal input   x(t)=X sin(wt)
The output y(t) can be found as follows;
substituting 'jw' for 's' in G(s) yields

$$G(jw)=\frac{K}{(jwT+1)}$$

The amplitude ration of the output to input is;

$$|G(jw)| = \frac{K}{\sqrt{1+T^2w^2}}$$

While the angle Ø = <G(jw)= - $tan^{-1}(wT)$

Thus , for the input x(t)=X sin(wt) , the output y(t) can be obtained from the following equation

$$y(t)= \frac{XK}{\sqrt{1+T^2w^2}} \sin(wt-tan^{-1}(wT))$$

Example: Consider the following transfer function

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{s^2 + 3s + 4}$$

Differential equation:

$$\frac{d^2c(t)}{dt^2} + 3\frac{dc(t)}{dt} + 4c(t) = r(t)$$

Sinusoidal transfer function

$$G(j\omega) = \frac{c(j\omega)}{s(j\omega)} = \frac{1}{(j\omega)^2 + 3(j\omega) + 4} = \frac{1}{4 - \omega^2 + 3j\omega}$$

BODE PLOT

Bode plot is a graphical representation of the transfer function for determining the stability of the control system.

Bode plot consists of two separated plots.

One is a plot of the logarithmic of magnitude of a sinusoidal transfer function, the other is a plot of the phase angle, both plots are plotted against the frequency. The curves are drawn on semilog graph paper using the log scale for frequency and linear scale for magnitude (in decibels) or phase angle (in degree) .

One advantage of using a semilog graph is that a large angle of frequency can be plotted and low frequency region is expanded.

Thus , Bode Plot consists of;

i) Magnitude in db = $20 \log |G(jw)|$ Vs log w

ii) Phase shift = $\emptyset$ Vs log w

The main advantage of using the Bode diagram

1. The magnitude multiplication of poles and zeros can be converted into addition and subtraction.
2. There is easy and simple sketching method to approximate the log-magnitude curve.
3. It depends on the asymptotic approximation.
4. If data of bode diagram are available , the transfer function of a system can be determined experimentally from frequency response.

The bode plot of the basic element of transfer function

1) The gain K

. If the value of gain K is greater than one , then the magnitude of K in dB is positive value while when the value of K is less than one the magnitude is negative.

. The constant gain K has a horizontal straight line log-magnitude curve with magnitude equal to 20 log K.

. The phase angle of the gain K is zero.

. If the gain of transfer function changed the entire curve of the log-magnitude either raises or lowers according to the new values of the K while the phase curve is not effected.

20 log (K x 10^n)= 20 log K + 20n

20 log K= -20 log$\frac{1}{K}$

2) Integral and derivative factors $(jw^{\mp})$

The logarithmic magnitude of 1/jw is ; $20 \log \left| \frac{1}{jw} \right| = -20 \log(w)$ **dB**

The phase angle of 1/jw is; constant and equal to - 90^o

The log-magnitude curve is a straight line with a slope of -*20* dB/decade

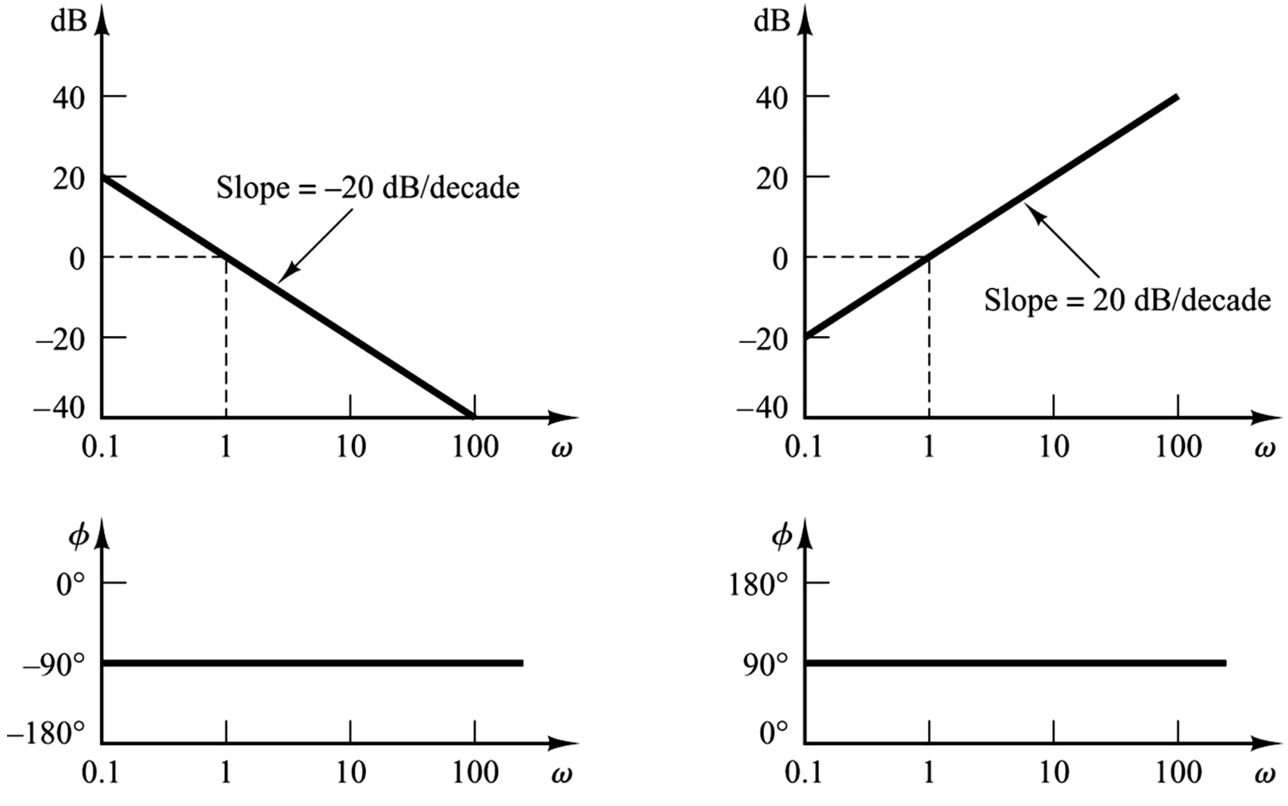
The logarithmic magnitude of jw is ; $20 \log \left| jw \right| = 20 \log(w)$ **dB**

The phase angle of jw is ; constant and equal to + 90^o

The log-magnitude curve is a straight line with a slope of +*20* dB/decade

Figure (a) shows the log-magnitude and phase shift of $\frac{1}{jw}$

Figure (a) shows the log-magnitude and phase shift of jw



Bode diagram of $G(j\omega) = 1/j\omega$

(a)

Bode diagram of $G(j\omega) = j\omega$

(b)

The case when transfer function contains the multiple factor $\frac{1}{(jw)^n}$ or $(jw)^n$

The log magnitude is $20 \log \left| \frac{1}{(jw)^n} \right| = -n \times 20 \log |jw| = -n \times 20 \log w$ dB

The slope of the log-magnitude curves is -20 x n dB/decade and the phase angle is equal to -90 x n over the entire frequency range. The magnitude pass through the point (0 dB, w=1)

The log magnitude is $20 \log |(jw)^n| = n \times 20 \log |jw| = n \times 20 \log w$ dB

The slope of the log-magnitude curves is 20 x n dB/decade and the phase angle is equal to 90 x n over the entire frequency range. The magnitude pass through the point (0 dB, w=1)

3) The first order factors $(1 + jwT)^{\pm 1}$

The log magnitude of first order pole $\frac{1}{(1+jwT)}$ = 20 log $\left|\frac{1}{1+jwT}\right| = -20 \log \sqrt{1 + w^2T^2}$ dB

For low frequencies , such that w « $\frac{1}{T}$, the log magnitude may be approximated by

-20 log $\sqrt{1 + w^2T^2} \approx$ -20log 1 =0 dB

Thus the log-magnitude curve at low frequencies is the constant 0 dB straight line or low frequency asymptote
The low frequency asymptote is a straight line start from zero frequency to point of intersection with high frequency asymptote, and coincide on the frequency axis.

For high frequencies , such that w » $\frac{1}{T}$, the log magnitude may be approximated by

-20 log $\sqrt{1 + w^2T^2} \approx$ -20log (wT) dB this is a straight line with slop -20 dB/decade or high frequency asymptote

Thus, in order to draw the high frequency asymptote, we need two points.
at ω = 1/T , the log magnitude equals 0 dB. The first point (1/T, 0 dB)
at ω = 10/T , the log magnitude is -20 dB. The second point (10/T , -20 dB)

For ω » 1/T, the log-magnitude curve is thus a straight line with a slope of -20 dB/decade.

Simlog graph paper

The phase angle of $\frac{1}{1+jwT}$

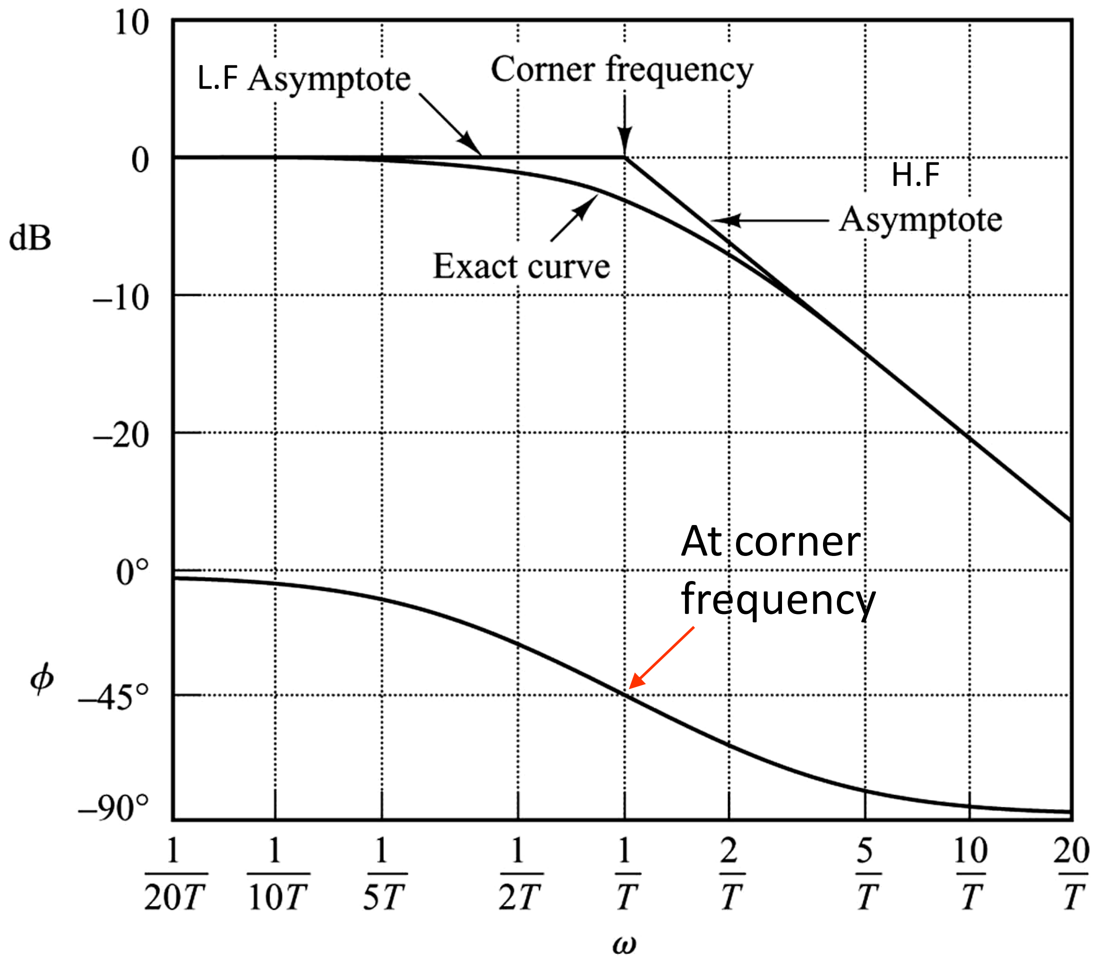
$\phi = -\tan^{-1}(wT)$

At w =0 , $\phi = 0^o$

At w = 1/T , $\phi = -\tan^{-1}(\frac{1}{T}\ T).$

$\phi = -45^o$

At w=∞ , $\phi = -90^o$



This figure shows the exact Log-magnitude curve, and its asymptotes, and phase-angle curve of 1/(1+*j*ωT).

The error between the exact magnitude curve and asymptotes can be computed as;

At w = $\frac{1}{T}$, the exact gives -20 log $\sqrt{1+w^2T^2}$ = -20 log $\sqrt{1+1}$ = -3.03 dB wh
the approximate -20 log wT = -20 log (1) = 0 dB

Error =-3.03 -0 = -3.03 dB

At w= $\frac{1}{2T}$, the exact gives -20 log $\sqrt{1+\frac{1}{4}}$ = -20 log ($\frac{\sqrt{5}}{2}$) = -0.97 dB
the approximate -20 log (1) *= 0 dB*

Error = -0.97- 0 = - 0.97

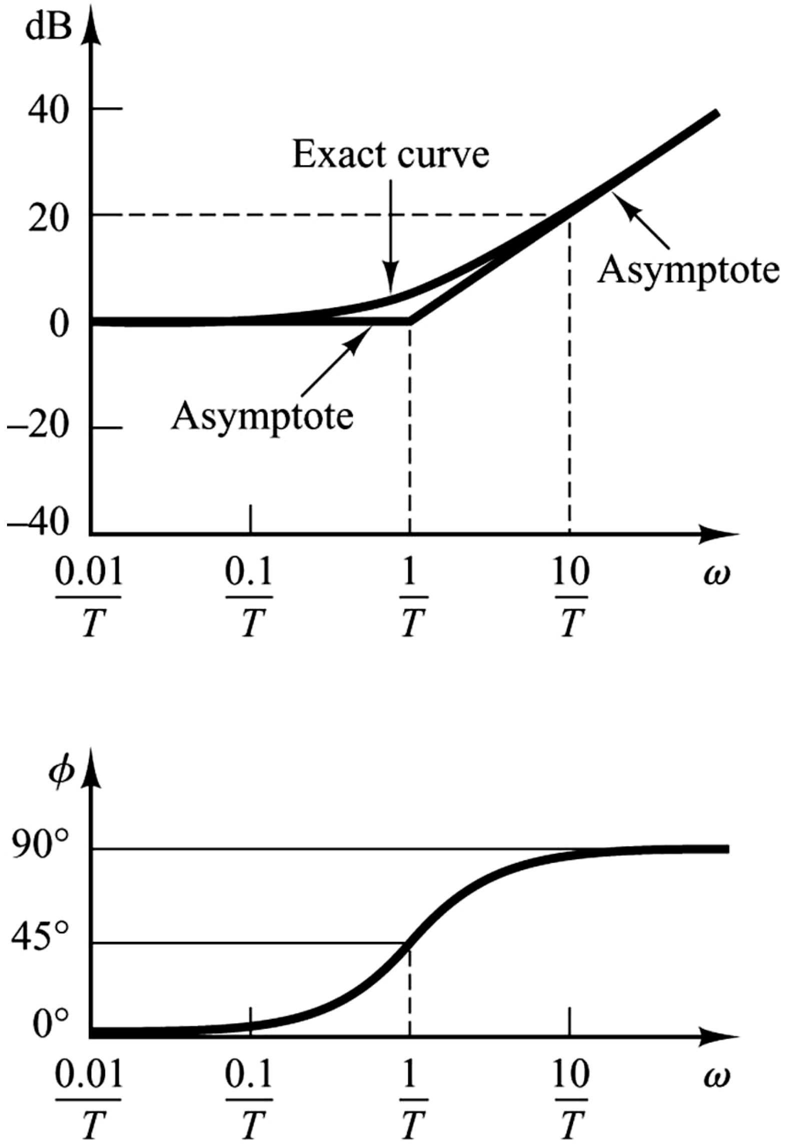
At w=$\frac{2}{T}$, the exact gives -20 log$\sqrt{1+4}$ = -20 log$\sqrt{5}$ =-6.9897 dB
, the approximate -20 log (2) = - 6.0205 dB

Error = -6.9897- 6.0205 = - 0.97 dB

The log magnitude of first order zero (1+jwT)= 20 log $|(1+jwT)| = 20\log\sqrt{1+w^2T^2}$ dB

and the phase shift $\phi = tan^{-1}(wT)$

The difference between the log-magnitude and the phase-angle curves of factor $1 + j\omega T$ (or zero) and the factor $\frac{1}{1+jwT}$ (or pole) is only sign.



This figure shows the exact Log-magnitude curve, and its asymptotes, and phase-angle curve of ($1+j\omega T$).

The case of multiple factors $(1 + jwT)^{\pm n}$

The exact log magnitude $\frac{1}{(1+jwT)^n}$ = 20 log $\left|\frac{1}{(1+jwT)^n}\right| = -20 \text{ x } n \text{ log} \sqrt{1 + w^2T^2}$ dB

The approximate log magnitude

At wT« 1 log magnitude = -20 x n log $\sqrt{1}$ = 0 dB low frequency asymptote

At wT » 1 log magnitude = -20 x n log (wT) high frequency asymptote

To correct at frequencies w= $\frac{1}{T}$, $\frac{1}{2T}$,and $\frac{2}{T}$

At all frequencies , the corrections are multiplied by *n*.

In the similar method the asymptotes construction may be made. The corner frequency is still at w=1/T, and the asymptotes are straight lines. The low-frequency asymptotes is a horizontal line while the high frequency asymptote has the slope of -20n dB/decade or 20n dB/decade

The exact phase angle

<G(jw) = ϕ = - n $tan^{-1}(wT)$

At w=0 ϕ = 0

At w= ∞ ϕ = -90 x n

At w= 1/T ϕ = (-90 x n)/2

Also, the difference between the log-magnitude and the phase-angle curves of multiple factors $1 + j\omega T$ (or zero) and the factor $\frac{1}{1+jwT}$ (or pole) is only sign

4) The quadratic Factors $[\,1 + 2\zeta\,(jw/w_n) + (\,jw/w_n)^2]^{\pm 1}$

If $\zeta \geq 1$ the factor can be represented as a product of two first order poles, and the bode diagram can be obtained as before.

If $0 < \zeta < 1$, the quadratic factors represent complex conjugate poles.

G(s) = $\frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}$

Put s=jw

G(jw) = $\frac{w_n^2}{(jw)^2+2\zeta w_n(jw)+w_n^2}$

G(jw) = $\frac{w_n^2}{-w^2+2\zeta w_n(jw)+w_n^2}$

G(jw) = $\frac{w_n^2}{w_n^2-w^2+j\,2\zeta w_n w}$

G(jw) = $\frac{1}{1-(\frac{w}{w_n})^2+j\,2\,\zeta\,\frac{w}{w_n}}$

20 log │G(jw)│=20 log│ $\frac{1}{1-(\frac{w}{w_n})^2+j\,2\,\zeta\,\frac{w}{w_n}}$ │

The exact magnitude in dB is ;

20 log │G(jw)│= -20 log $\sqrt{[1-\left(\frac{w}{w_n}\right)^2]^2+4\zeta^2(\frac{w}{w_n})^2}$

The approximate log magnitude is;

At low frequencies w « w_n log magnitude = - 20 log $\sqrt{1}$ = 0 dB low frequency asymptote

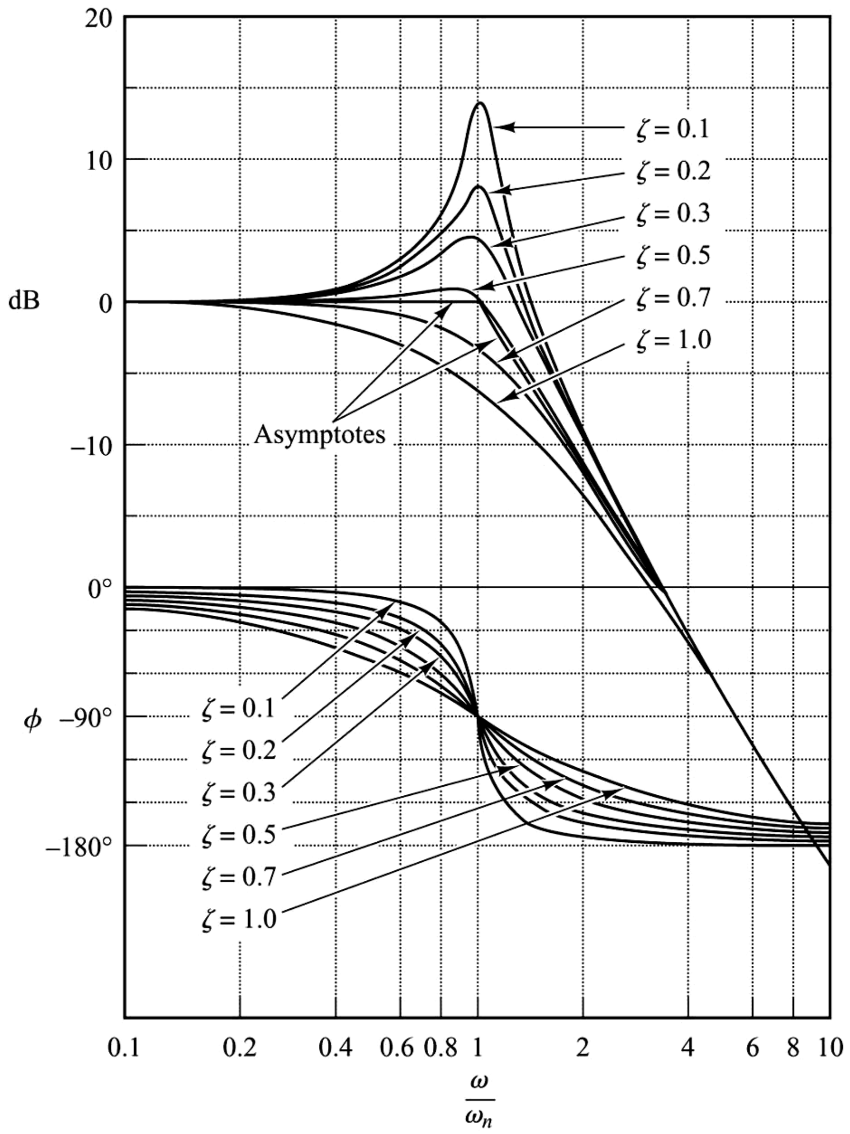
At high frequencies w » w_n log magnitude = - 20 log $\sqrt{\frac{w^4}{w_n^4}}$ = -20 log $(\frac{w}{w_n})^2$ = - 40 log $\frac{w}{w_n}$ high frequency asymptote

The high frequency asymptote intersect the low frequency asymptote at w = w_n

Since at this frequency - 40 log $\frac{w}{w_n}$ =- 40 log (1) = 0 dB

Thus, this frequency is the corner frequency for the quadratic poles.
The two a asymptotes just derived are independent of the value of ζ.

Near the frequency w = w_n , a resonant peak occurs. The damping ratio ζ determines the magnitude of this resonant peak. Error obviously exist in the approximation by straight line asymptotes. The magnitude of the error depends on the value of ζ . It is large for small values of ζ .



The figure shows the exact Log-magnitude and phase shift curves and the asymptotes of the quadratic transfer function

The exact phase angle of quadratic factor $[1 + 2\zeta(jw/w_n) + (jw/w_n)^2]^{-1}$ is ;

$$<G(jw) = \phi = -\tan^{-1}\left(\frac{2\zeta\frac{w}{w_n}}{1-(\frac{w}{w_n})^2}\right)$$

The phase angle is a function of both w and ζ. At w = 0 , the phase angle equals 0^o. A

At the corner frequency w = w_n , the phase angle is -90^o regardless of ζ ,

$$\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}(\infty) = -90^o$$

At w= ∞ , the phase angle becomes -180^o. The phase angle curve is skew symmetric about the inflection point, the point where ϕ= -90^o. There are no simple ways to sketch such phase curve.

The frequency response curves of the factor (or complex conjugate zeros) $[1 + 2\zeta(jw/w_n) + (jw/w_n)^2]$ can be obtained by merely reversing the sign of the log magnitude and that of the phase angle of the factor