Nonlinear Systems

Fourth grade-Control Engineering **CSE-C4307**

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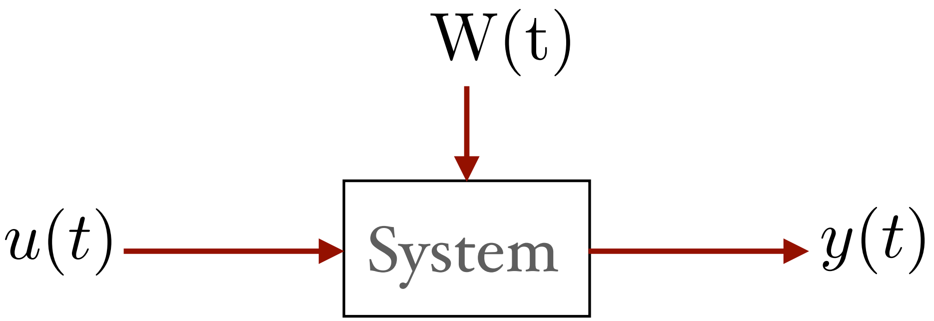
What Is a System?



- Set of elements
- Set of relations
- Functioning together

- **A system***: is an object in which variables of different kinds interact and produce observable signals: *Outputs*.
- Its external signals are either *Inputs* or *Disturbances*.

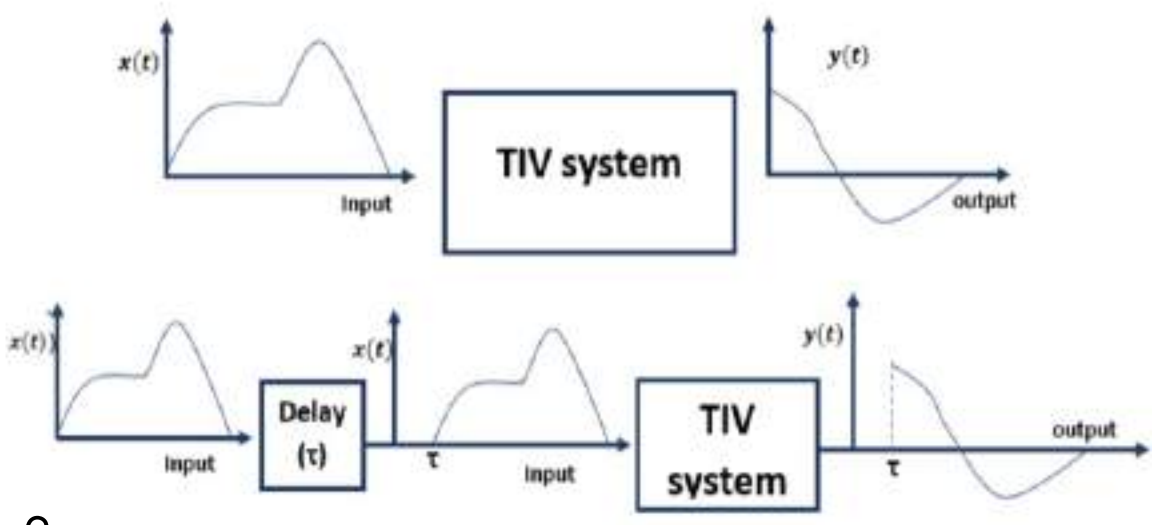
*Taken from your System Identification course lecture 1

There Two different types of systems: Linear and Nonlinear

Linear Time-Invariant Systems

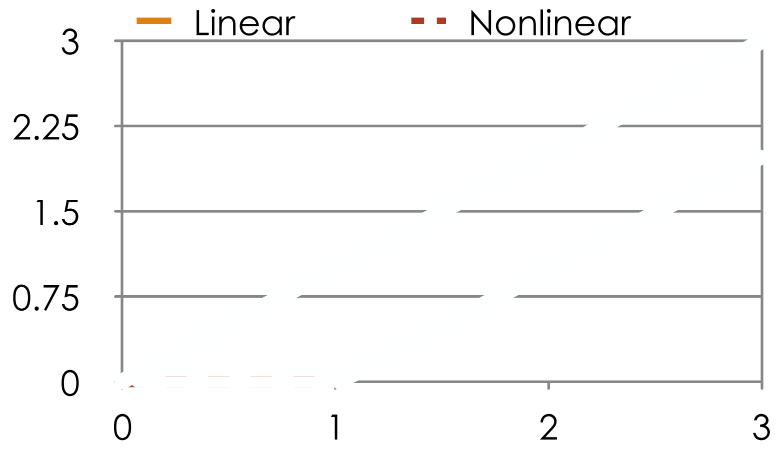
NOT changing over TIME

Superposition Principle applies



What is a Nonlinear System??

- A nonlinear system is a system in which the change of the output is not proportional to the change of the input.
- Nonlinear problems are of interest because most systems are inherently nonlinear in nature.
- For linear systems, the controlled response is much more predictable. But when you close a control loop around a nonlinear system, unpredictable and chaotic behavior is expected.



Nonlinear Systems [1]

The subject of nonlinear control deals with the analysis and the design of nonlinear control systems i.e., of control systems containing at least one nonlinear component.

In the analysis, a nonlinear closed-loop system is assumed to have been designed, and we wish to determine the characteristics of the system's behavior.

A system is nonlinear if the principle of Superposition <u>does not apply.</u>

What is a Nonlinear System??

A system is nonlinear if the principle of Superposition does not apply.

(Superposition is a Necessary and sufficient Condition for linearity).

A system is NONLINEAR if the response to 2 or more I/Ps cannot be calculated by treating each I/P separately and adding the results.

Superposition theorem: A system is *linear* if it is complied with the following mathematical properties:

1. Additivity
2. Homogeneity

Properties of Superposition [2]

1. **Additivity:** a signal in a linear system can be broken and processed individually and then it could be united again

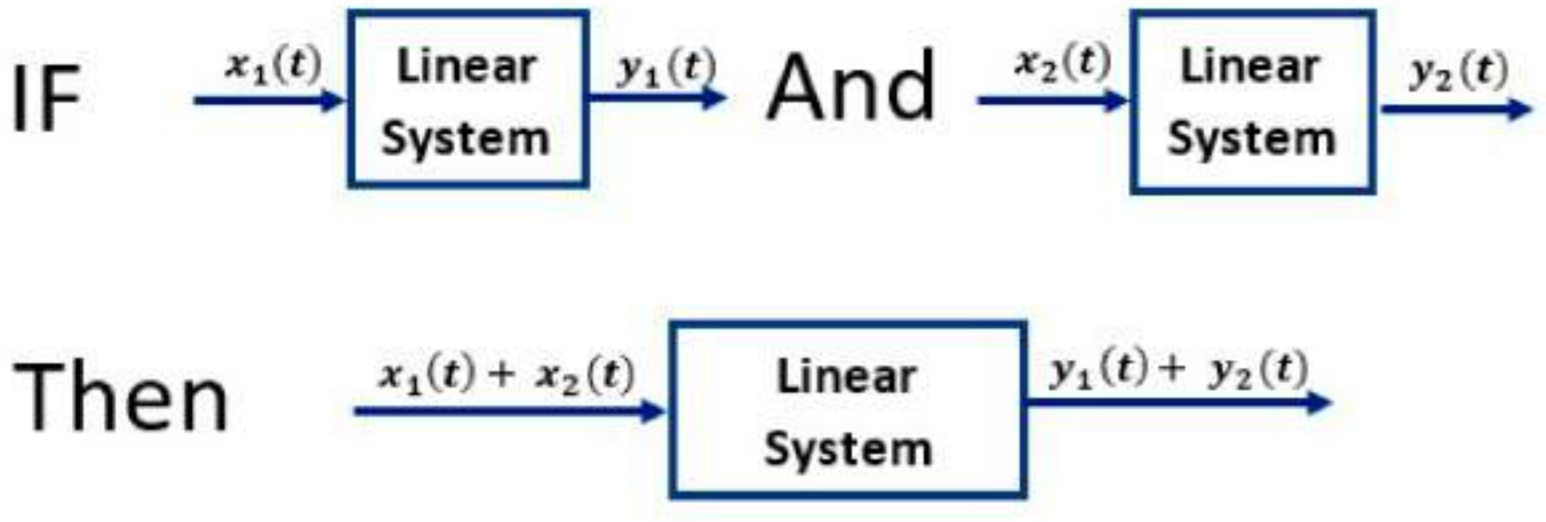


Figure 1: Additivity property of superposition theorem

Properties of Superposition [2]

2. **Homogeneity:** The change in the input's signal amplitude results in a corresponding change in the output signal amplitude.

Figure 2: Homogeneity property of superposition theorem

Examples [2]

Ex.1: $y(t) = x(\sin(t))$

LOA

$$x_1(t) \rightarrow sys \rightarrow y_1(t) = x_1(\sin(t))$$

$$x_2(t) \rightarrow sys \rightarrow y_2(t) = x_2(\sin(t))$$

$$\boxed{y_1(t) + y_2(t) = x_1(\sin(t)) + x_2(\sin(t)) \qquad ...(1)}$$

$$\boxed{x_1(t) + x_2(t) \rightarrow sys \rightarrow y(x_1(t) + x_2(t)) = x_1(\sin(t)) + x_2(\sin(t)) \qquad ...(2)}$$

$$(1) = (2)$$

LOH

$$ky(t) = kx(\sin(t)) \qquad ...(3)$$

$$y(kx(t)) = kx(\sin(t)) \qquad ...(4)$$

$$(3) = (4)$$

LOA and LOH are satisfied then the system is LINEAR

Examples [2]

Ex.2: $y(t) = x^2(t)$

LOA

$$y_1(t) = x_1^2(t)$$

$$y_2(t) = x_2^2(t)$$

$$y_1(t) + y_2(t) = x_1^2(t) + x_2^2(t) .....(1)$$

$$y(x_1(t) + x_2(t)) = (x_1(t) + x_2(t))^2 .....(2)$$

$$\Longrightarrow \ (1) \neq (2)$$

No need to verify for LOH (LOA and LOH must both be TRUE)

Then the system is NONLINEAR

Examples [2]

Ex.3 $y(t) = 2t + x(t)$

LOA

$$x_1(t) \to sys \to y_1(t) = 2t + x_1(t)$$

$$x_2(t) \to sys \to y_2(t) = 2t + x_2(t)$$

$$y_1(t) + y_2(t) = 4t + x_1(t) + x_2(t) \quad ...(1)$$

$$x_1(t) + x_2(t) \to sys \to y(t) = 2t + x_1(t) + x_2(t) \quad ...(2)$$

$$(1) \neq (2)$$

LOA is violated then system is NONLINEAR

HomeWork: check Linearity of the following systems:

$1)\, y(t) = \sin t \quad x(t)$

$2)\, y(t) = e^3 x(t)$

$3)\, y(t) = 3\log t - \sin t \quad x(t^2)$

$4)\, y(t) = x(t+1) + x(t-1)$

Nonlinear Ordinary Differential Equations

$f(t, y, y', \dots, y^{(n)})$ **is LINEAR ODE if it can be written as follows**

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = b(t) \quad \dots(**)$$

where $a_0(t), \dots, a_n(t), b(t)$ and are arbitrary differentiable functions that do not need to be linear, $y^{(n)}, \dots, y'$ are the successive derivatives of $y(t)$. .

Such that:

1. All of the <u>dependent variable</u> $y(t)$ and its derivatives are to the first power.

2. All of $y(t)$ and its derivatives DO NOT appear in a nonlinear function like

$$y^2(t), \sqrt{y(t)}, \sin y(t) \dots etc$$

3. All Coefficients of $y(t)$ and its derivatives are PURE functions of the independent variable t

Nonlinear Ordinary Differential Equations

Is the following ODE linear?

1. $y'' - 5ty = ty' - 25, \quad y = f(t)$

2. $\left(\dfrac{dy}{dt}\right)^2 - 2t^2 = 0$

<u>Sol.</u> 1) Rearrange first equation to see if it matches ODE form in (**)

$$y''(t) - ty'(t) - 5ty(t) = -25$$

Thus the first equation is LINEAR.

2) The first term of the second equation is the square of the differential coefficient, hence it is NONLINEAR

$$\left(\frac{dy}{dt}\right)^2 - 2t^2 = 0$$

Nonlinear ODE System

H.W: Check the following ODE for Linearity

1. $\ddot{y}(t) + 3y(t)\ddot{y}(t) + y(t) = e^{t}$

2. $\ddot{y}(x) + e^{x}\dot{y}(x) + y(x) = \sin x$

3. $\sqrt{\dfrac{dy(x)}{dx}} + y(x) = 7\cos x$

4. $y^{(2)}(t)y(t) = \log x$

System of ODE

A number of coupled differential equations form a system of equations.

The focus will be here on the system of finite number of first order ODE:

$$\begin{pmatrix} y_1' \\ \cdot \\ \cdot \\ y_n' \end{pmatrix} = \begin{pmatrix} f_1(t,Y) \\ \cdot \\ \cdot \\ f_m(t,Y) \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

If $f_1(t,Y),\ldots, f_m(t,Y)$ have ***one or more nonlinear terms***, then the system is ***nonlinear.***

System of Linear ODE

Example1: The following system is nonlinear since it has two nonlinear terms:

$$y_1' = y_1 + \boxed{y_1 y_2}$$
$$y_2' = y_1 + \boxed{y_2^3}$$

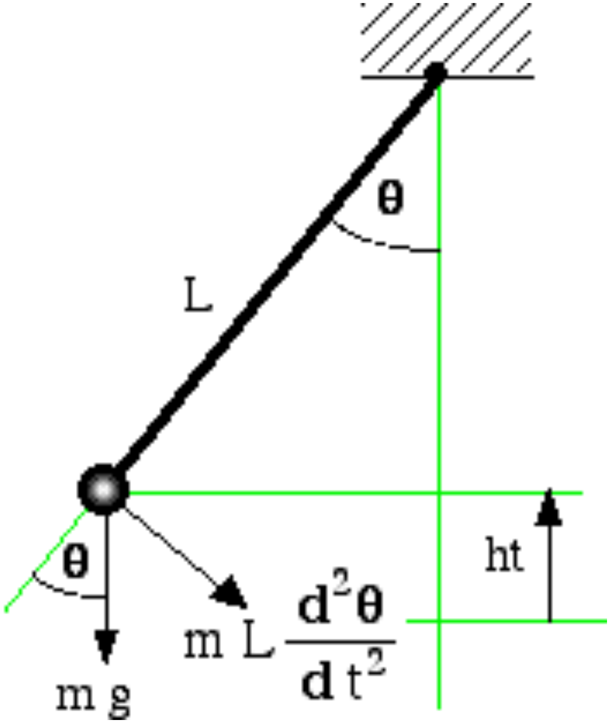
Nonlinear Systems Examples

Frictionless PENDULUM Equation

From a force balance we obtain:

$$m\,g\,\sin(\theta) + m\,L\,\frac{d^2\theta}{dt^2} = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$



$$let \quad y(t) = \theta(t)$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\tfrac{g}{L}\sin y$$

It is a <u>Nonlinear</u> System

Nonlinear Systems Examples

Minimal model of glucose kinetics of Type 1 Diabetes: (BERGMAN minimal model)

$$\dot{G}(t) = -p_1.(G(t) - G_b) - X(t).G(t)$$
$$\dot{X}(t) = -p_2.X(t) + p_3.(I(t) - I_b)$$
$$\dot{I}(t) = p_4.(G(t) - p_5)^+.t - p_6(I(t) - I_b)$$

$$(G(t) - p_5)^+ = \begin{cases} 0 & if \quad G \le p_5 \\ (G - p_5) & if \quad G > p_5 \end{cases}$$

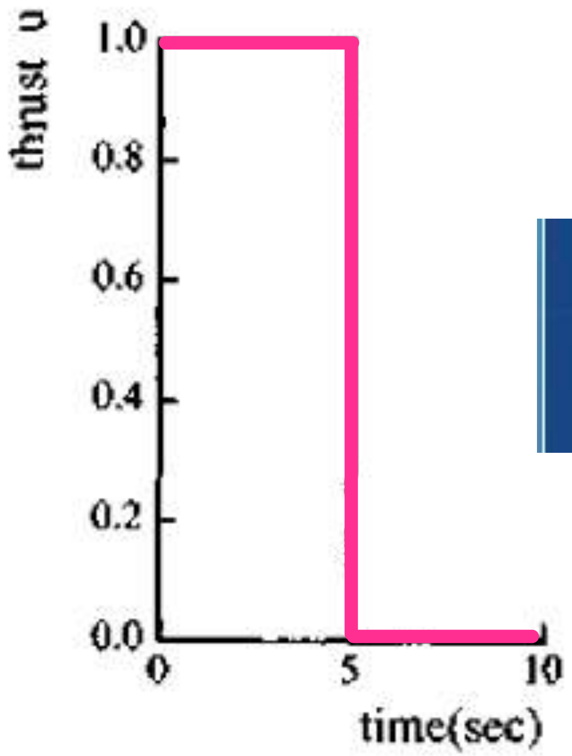
$\boxed{G(t), X(t), I(t)}$ are Glucose and interstitial and plasma insulin respectively.

$G_b, I_b, p_i, \quad i = 1,..5$ are constants.

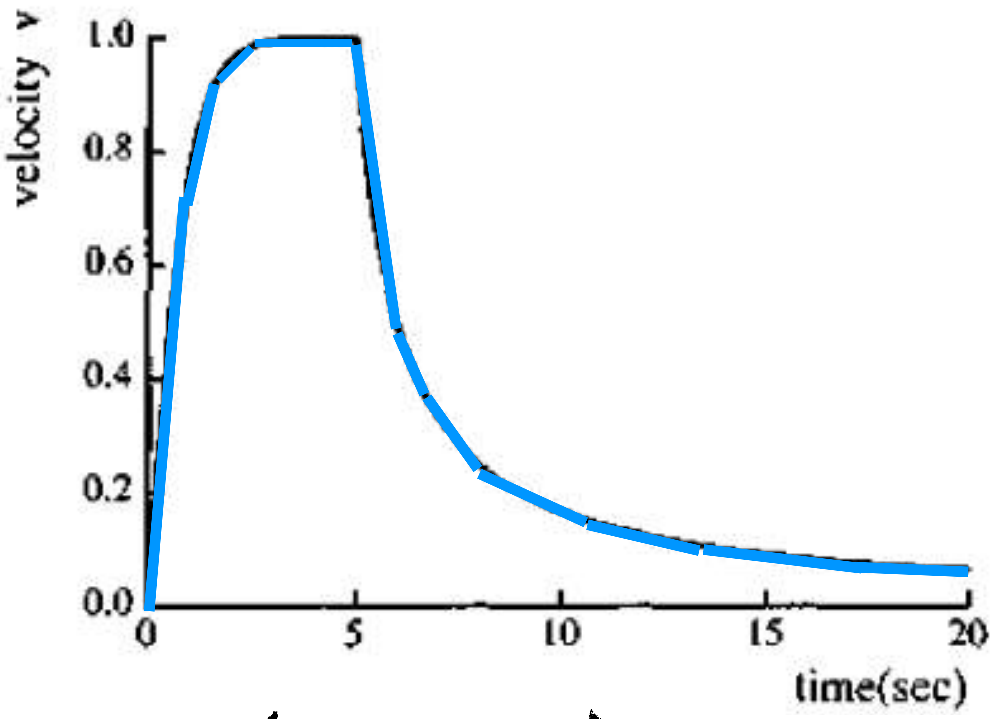
Nonlinear Systems Behavior

Example 1.1 of [1]: A simplified model of the motion of an underwater vehicle can be written

$$\dot{v} + |v|v = u$$



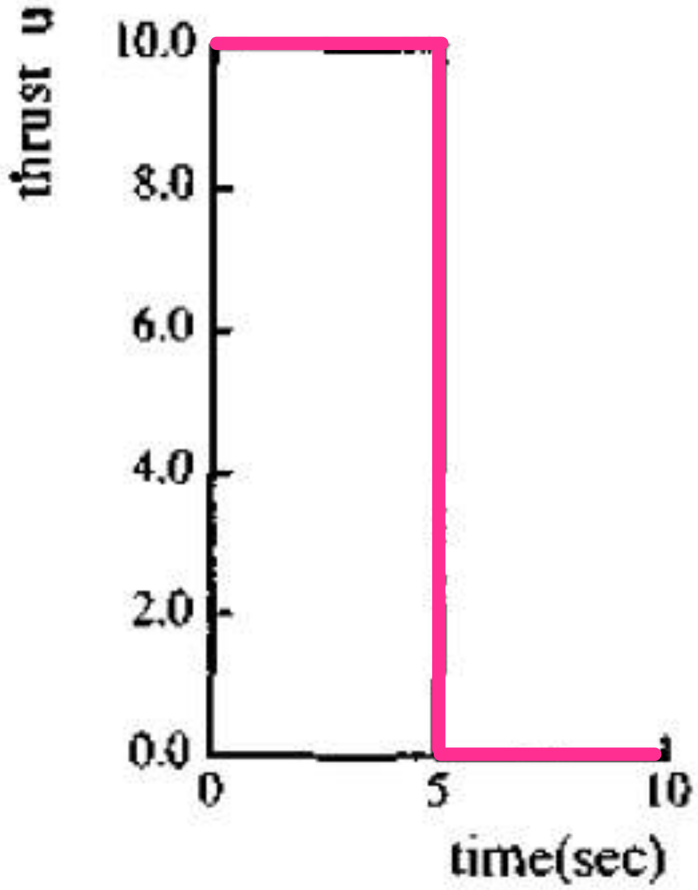
https://www.ecagroup.com/en/solutions/a9-e-auv-autonomous-underwater-vehicle

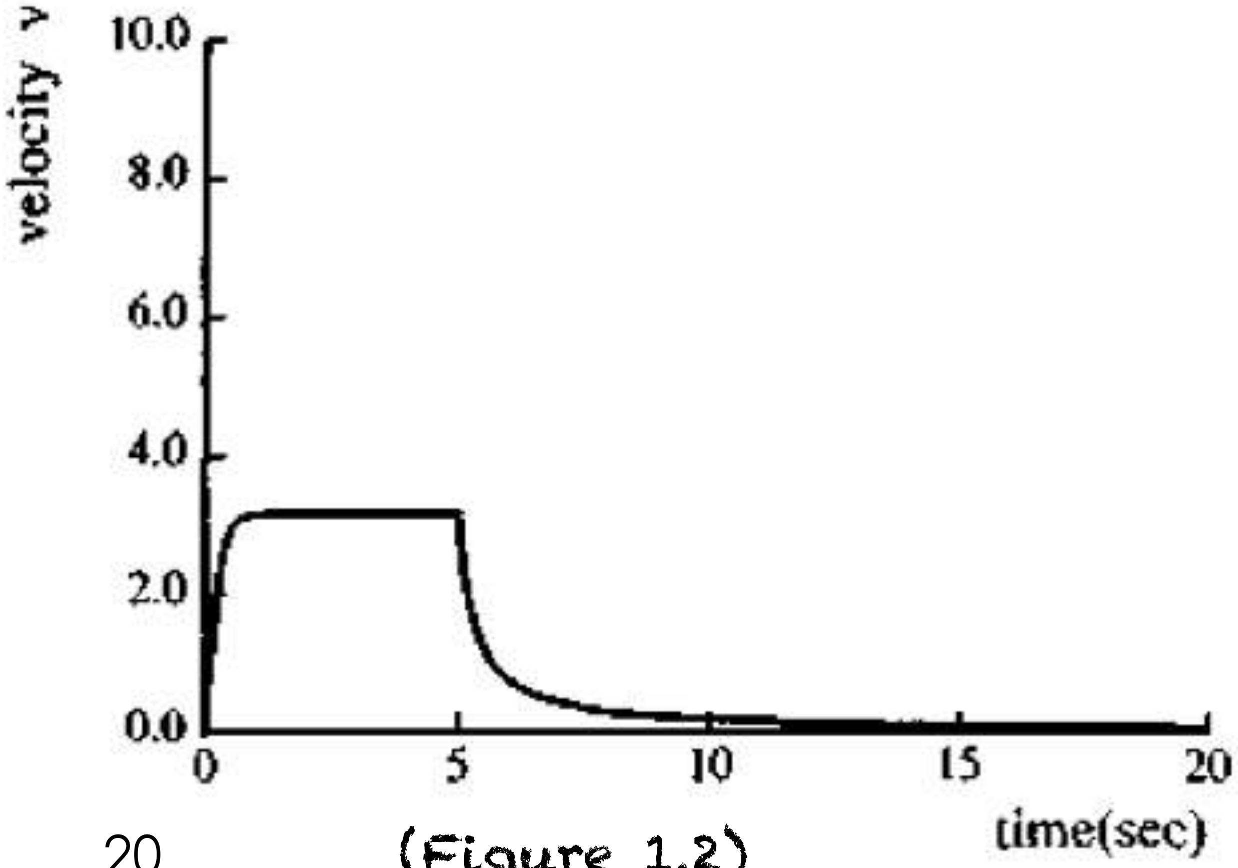


(Figure 1.1)

Example 1.1 of [1]..ctd

Applying u=10 on the system: $\dot{v} + |v|v = u$





(Figure 1.2)

Example 1.1 of [1]..ctd

$$\dot{v} + |v|v = u \qquad (1)$$

- The settling speed (v_s) in response to the first step is <u>not 10 times</u> that obtained in response to the first unit step in the first experiment, as it would be in a linear system.

- This can be understood intuitively, by setting steady state in equation (1):

$$u = 1 \implies 0 + |v_s|v_s = 1 \implies v_s = 1$$

$$u = 10 \implies 0 + |v_s|v_s = 10 \implies v_s = \sqrt{10} \approx 3.2$$

- Carefully understanding and effectively controlling this nonlinear behavior is particularly important if the vehicle is to move in a large dynamic range and change speeds continually, as is typical of industrial remotely-operated underwater vehicles.

NONLINEAR Systems [1]

-Physical systems are inherently nonlinear. Thus, all control systems are <u>nonlinear</u> to a certain extent.

-However, if the operating range of a control system is small, and if the involved nonlinearities are smooth, then the control system may be reasonably approximated by a linearized system. Its dynamics is described by a set of linear differential equations.

Ex. Pendulum equation, if the range of the movement is small (small angle Approximation): $\frac{d^2\theta}{dt^2} = \frac{-g}{l} sin\theta \quad if \quad \theta \approx 0 \quad \Longrightarrow \quad sin\theta = \theta \quad \Longrightarrow \quad \frac{d^2\theta}{dt^2} = \frac{-g}{l}\theta$

WHY Nonlinear Control? [1]

1. <u>Improvement of existing control systems:</u>

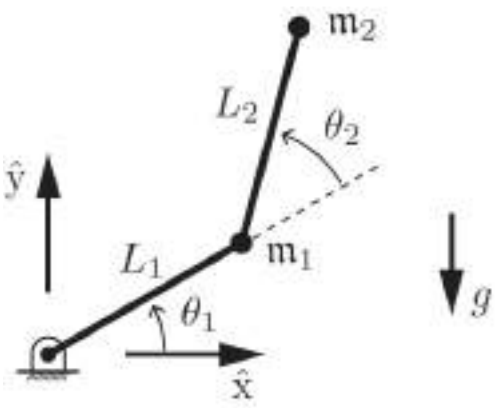
- Linear control methods rely on the key assumption of small range operation for the linear model to be valid.
- When the required operation range is large, a <u>linear controller</u> is likely to perform very poorly or to be unstable, because the nonlinearities in the system cannot be properly compensated for.
- Nonlinear controllers, on the other hand, may handle the nonlinearities in large range operation directly.

WHY Nonlinear Control? [1]

1. **Improvement of existing control systems (Robot Example):**

- This point is easily demonstrated in robot motion control problems.
- When a linear controller is used to control robot motion, it neglects the nonlinear forces associated with the motion of the robot links.
- The controller's accuracy thus quickly degrades as the speed of motion increases, because many of the dynamic forces involved, such as Coriolis and centripetal forces, vary as the square of the speed.

1. 2DOF Robot example[3]:



$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta),$$

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$c(\theta, \dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1 + \theta_2) \end{bmatrix},$$

where $M(\theta)$ is the symmetric positive-definite mass matrix, $c(\theta, \dot{\theta})$ is the vector containing the Coriolis and centripetal torques, and $g(\theta)$ is the vector containing the gravitational torques. These reveal that the equations of motion are linear in $\ddot{\theta}$, quadratic in $\dot{\theta}$, and trigonometric in θ.

Quadratic terms containing $\dot{\theta}_i^2$ are called **centripetal** terms, and quadratic terms containing $\dot{\theta}_i\dot{\theta}_j, i \neq j$, are called **Coriolis** terms.

WHY Nonlinear Control? [1]

1. **Improvement of existing control systems (Robot example):**

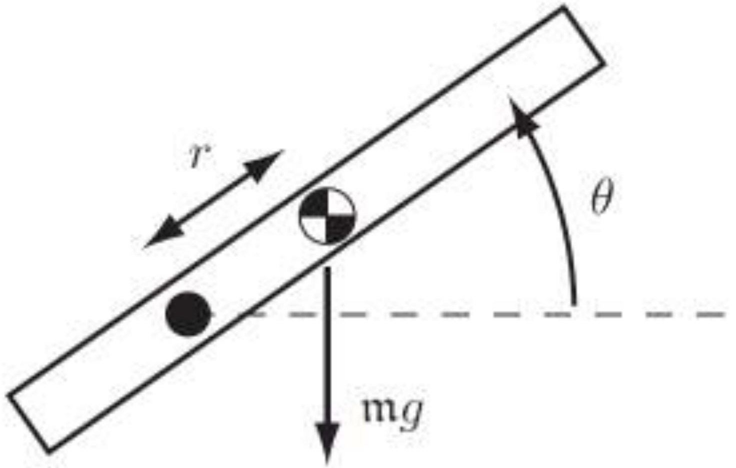
- Therefore, in order to achieve a pre-specified accuracy in robot tasks such as pick-and-place, arc welding and laser cutting, the speed of robot motion, and thus productivity, has to be kept low.
- On the other hand, a conceptually simple nonlinear controller, commonly called computed torque controller, can fully compensate the nonlinear forces in the robot motion and lead to high accuracy control for a very large range of robot speeds and a large workspace.

WHY Nonlinear Control?

Example: A single-joint robot rotating under gravity [3]

Consider a single motor attached to a single link, as shown. Let τ be the motor's torque and θ be the angle of the link. The dynamics with the friction torque can be written as

$$\tau = M\ddot{\theta} + \underbrace{\mathfrak{m}gr\cos\theta + b\dot{\theta}}_{h(\theta,\dot{\theta})}, \qquad b > 0.$$



where M is the scalar inertia of the link about the axis of rotation, $\mathfrak{m}$ is the mass of the link, r is the distance from the axis to the center of mass of the link, and $g \geq 0$ is the gravitational acceleration.

h contains all terms that depend only on the state, not the acceleration.

Example: A single-joint robot rotating under gravity[3] pp. 429-430

Let's combine PID control with a model of the robot dynamics $\{\tilde{M}, \tilde{h}\}$ to achieve the error dynamics

$$\ddot{\theta}_e + K_d\dot{\theta}_e + K_p\theta_e + K_i\int\theta_e(t)\,dt = 0$$

$$\ddot{\theta}_e = \ddot{\theta}_d - \ddot{\theta}$$

$$\ddot{\theta} = \ddot{\theta}_d + K_d\dot{\theta}_e + K_p\theta_e + K_i\int\theta_e(t)\,dt \qquad (*)$$

suppose we have a model of the robot arm as follows

$$\tau = \tilde{M}\ddot{\theta} + \tilde{h}(\dot{\theta},\theta) \qquad (**)$$

substitute (*) in eq. (**) yields:

$$\boxed{\tau = \tilde{M}\left(\ddot{\theta}_d + K_p\theta_e + K_i\int\theta_e(\mathrm{t})dt + K_d\dot{\theta}_e\right) + \tilde{h}(\theta,\dot{\theta}).}$$

Example: Robotic Arm with computed torque control [3] pp. 429-433

A conceptually simple nonlinear controller, commonly called *computed torque controller*, can fully compensate the nonlinear forces in the robot motion and lead to control for a very large range of robot speeds and a large workspace.

$$\tau = \tilde{M}\left(\ddot{\theta}_d + K_p\theta_e + K_i\int\theta_e(t)dt + K_d\dot{\theta}_e\right) + \tilde{h}(\theta,\dot{\theta}).$$

Computed Torque Control

$$\tau = M\ddot{\theta} + h(\theta,\dot{\theta})$$

Actual Robot arm

$$\ddot{\theta}_e + K_d\dot{\theta}_e + K_p\theta_e + K_i\int\theta_e(t)\,dt = 0$$

linear error dynamics

WHY Nonlinear Control? [1]

2. Analysis of hard nonlinearities:

-In control systems there are many nonlinearities whose discontinuous nature does not allow linear approximation. These so-called "***hard nonlinearities***" include Coulomb friction, saturation, dead-zones, backlash, and hysteresis, and are often found in control engineering.

-Nonlinear analysis techniques must be developed to predict a system's performance in the presence of these inherent nonlinearities.

-Because such nonlinearities frequently cause undesirable behavior like instabilities, their effects must be predicted and properly compensated for.

WHY Nonlinear Control? [1]

- There may be other reasons to use nonlinear control techniques, such as cost and performance optimality. As for performance optimality, we can cite bang-bang type controllers, which can produce fast response, but are inherently nonlinear.
- In the past, the application of nonlinear control methods had been limited by the computational difficulty associated with nonlinear control design and analysis.
- In recent years, however, advances in computer technology have greatly relieved this problem.

More details can be found in Section 1.1 in [1].

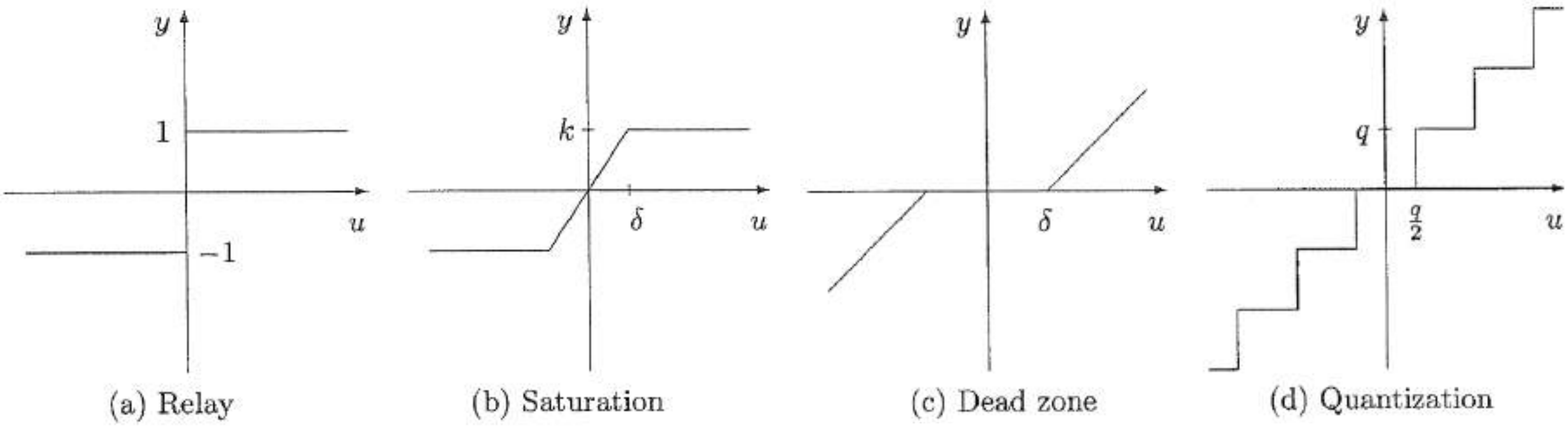
Types of Nonlinearities [1]

- **Nonlinearities can be classified as inherent *(natural) and intentional (artificial)*.**
- ***Inherent nonlinearities*** are those which naturally come with the system's hardware and motion.
- Examples of inherent nonlinearities include centripetal forces in rotational motion, and Coulomb friction between contacting surfaces.
- Usually, such nonlinearities have undesirable effects, and control systems have to properly compensate for them.
- ***Intentional nonlinearities*** are artificially introduced by the designer. Nonlinear control laws, such as adaptive control laws and bang-bang optimal control laws, are typical examples of intentional nonlinearities.

Types of Nonlinearities [1]

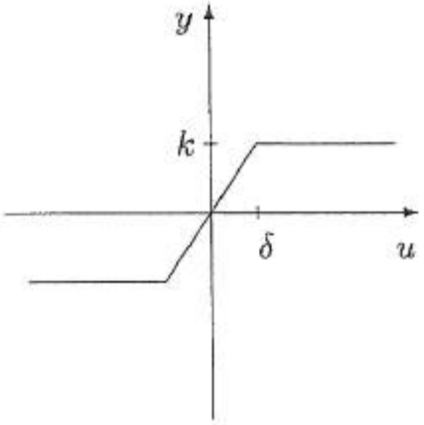
- **Nonlinearities can also be classified in terms of their mathematical properties, as *continuous and discontinuous*.**
- ***Discontinuous nonlinearities*** are also called ”hard“ nonlinearities because they cannot be locally approximated by linear functions.
- Hard nonlinearities (such as, e.g., backlash, hysteresis) are commonly found in control systems, both in small range operation and large range operation.
- Whether a system in small range operation should be regarded as nonlinear or linear depends on the magnitude of the hard nonlinearities and on the extent of their effects on the system performance.

Common hard Nonlinearities [4]



Common Nonlinearities [4]

- ***Saturation*** characteristics are common in all practical amplifiers (electronic, pneumatic, etc) and motors. They are also used intentionally as limiters.

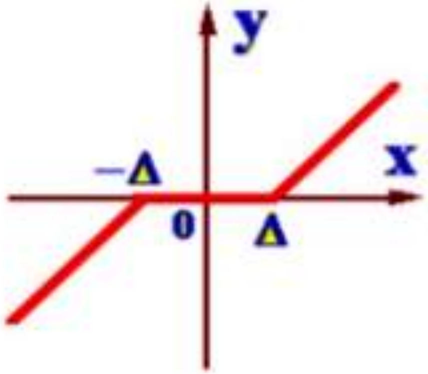


(b) Saturation

$$\mathrm{sat}(u) = \begin{cases} u, & \text{if } |u| \le 1 \\ \mathrm{sgn}(u), & \text{if } |u| > 1 \end{cases}$$

-***Dead-zone*** nonlinearity is typical in valves and some amplifiers at low input signals.

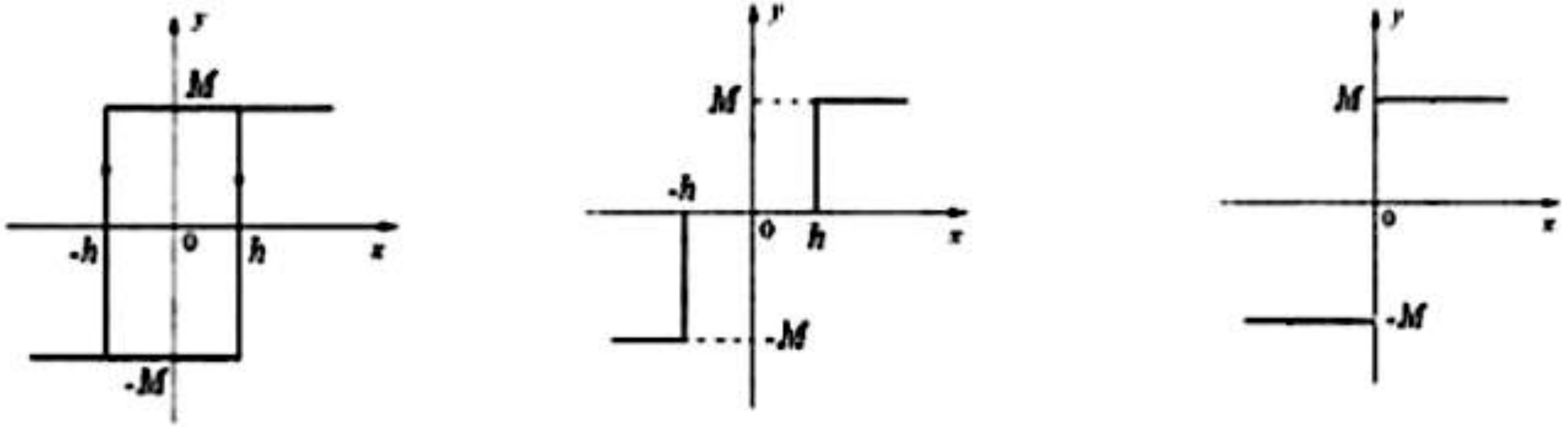
$$y(t) = \begin{cases} 0 & |x(t)| \le \Delta \\ K[x(t) - \Delta \cdot \mathrm{sgn}\, x(t)] & |x(t)| > \Delta \end{cases}$$



(c) Dead zone

Relay Nonlinearity[*]

- ***Relay:*** is an intentional nonlinearity. Ideal relay is the extreme case of saturation.
- occurs when the slope in linearity range becomes vertical .
- It can lead to chattering due to discontinuity.
- Relay coil require a finite amount of current to actuate (cause of dead zone).
- Relay characteristic can exhibit hysteresis.



- [*] https://www.slideshare.net/nidaunapprochablesta/types-of-nonlinearities

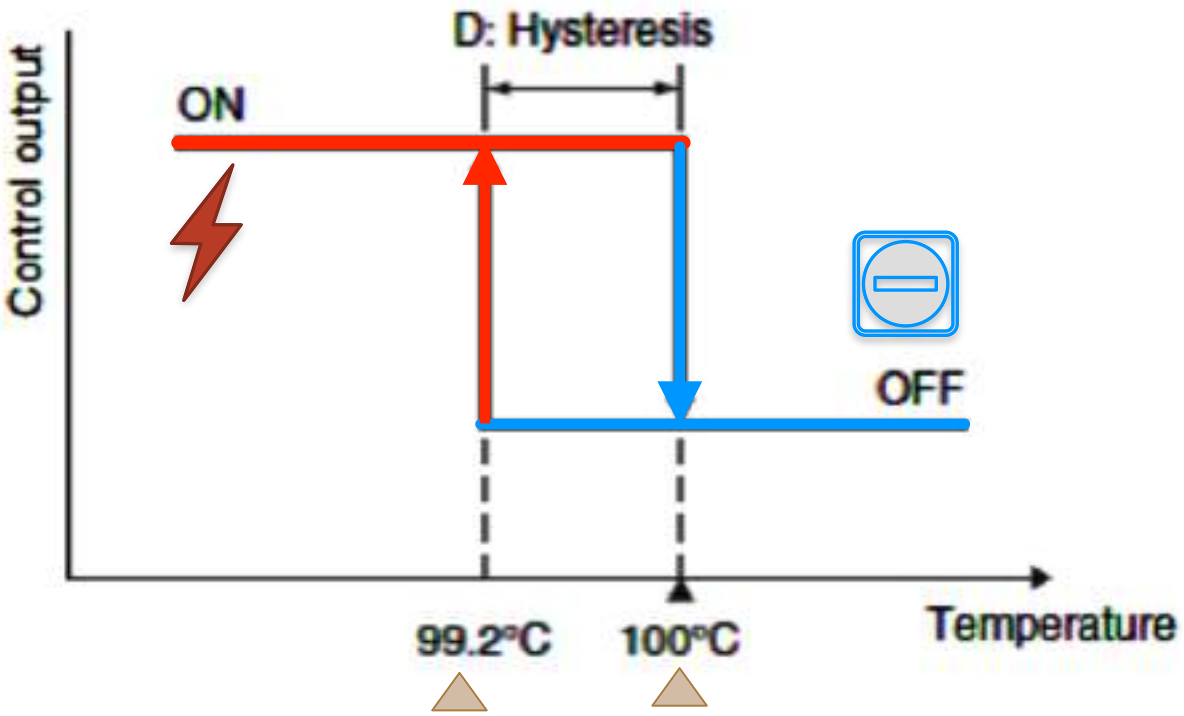
Temperature Control: Hysteresis[*]

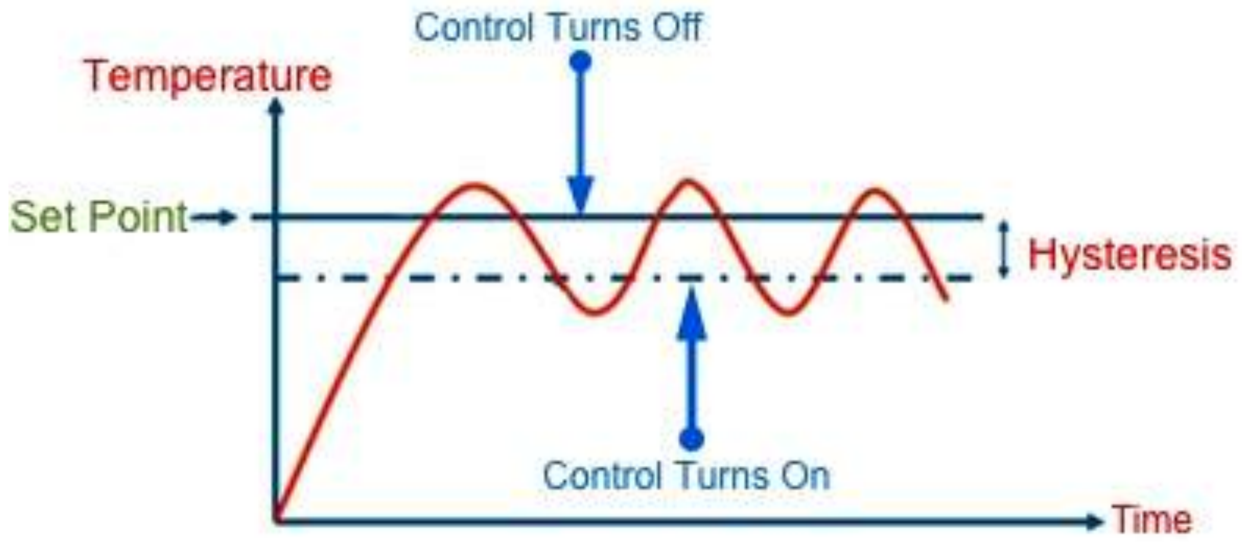
- ON/OFF controller switches the actuator ON or OFF based on the set point.
- The output frequently changes according to minute temperature changes.
- This shortens the life of the output relay or unfavorably affects some devices connected to the Temperature Controller.
- To prevent this from happening, a temperature *band called hysteresis* is created between the ON and OFF operations.

[*] http://www.omron-ap.com/service_support/FAQ/FAQ00549/index.asp

Temperature Control: Hysteresis[*]

Example: If a Temperature Controller with a temperature range of (0 to 400°C) has a 0.2% hysteresis, (D) in the figure will be 0.8°C. If the set point is 100°C, the output will turn OFF at a process value of 100°C and will turn ON at a process value of 99.2°C.





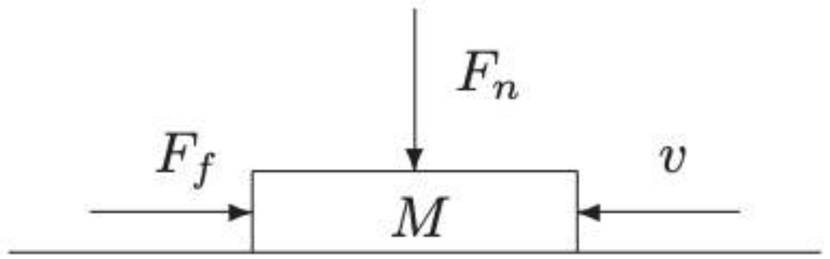
[*] http://www.omron-ap.com/service_support/FAQ/FAQ00549/index.asp

Temp response figure from https://blog.belilove.com/2016/06/on-off-temperature-control-using-plc.html

Common hard Nonlinearities

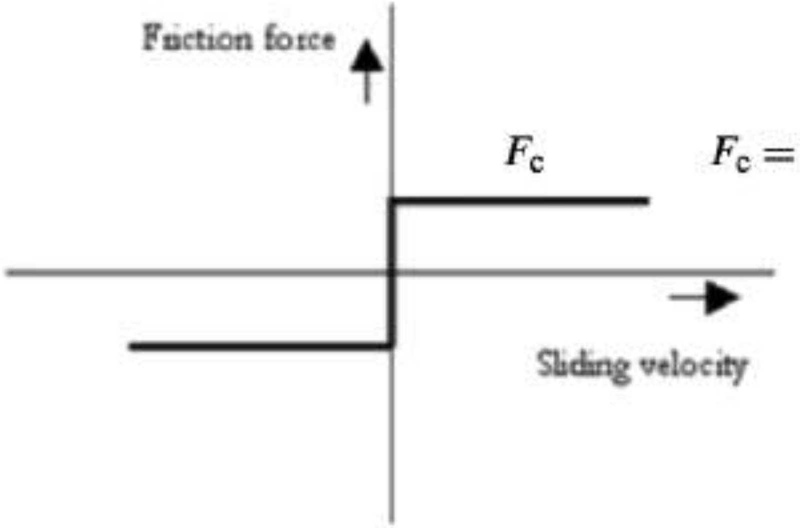
- ***Friction:***

where F_n is the normal force, μ the friction coefficient and v the relative velocity of the moving object. This is schematically represented in figure 1.2.



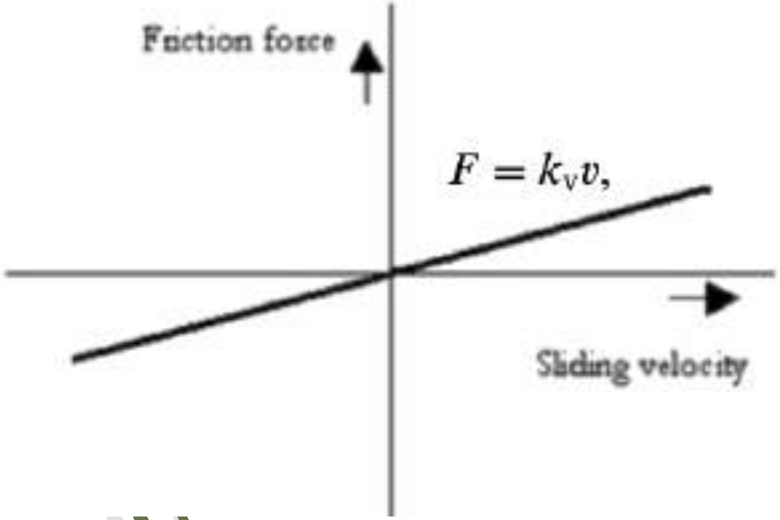
- Static Friction

- Coulomb Friction

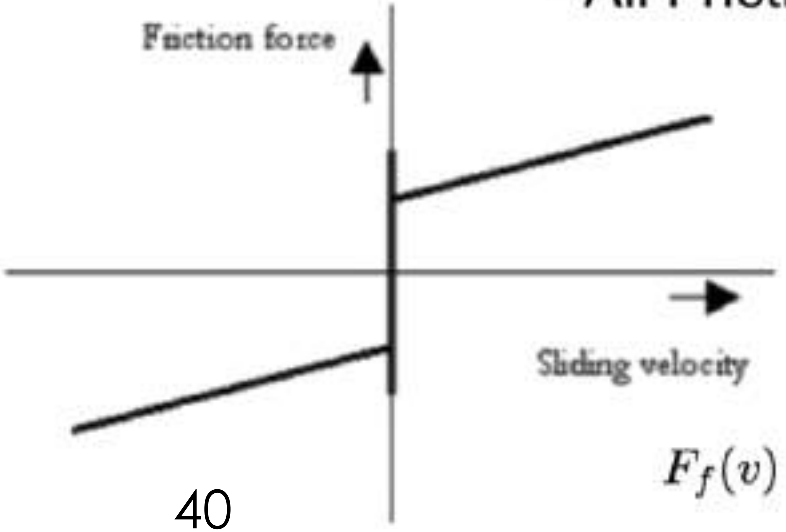


$$F_c = \begin{cases} -\mu_k mg, & \text{for} \quad v < 0 \\ \mu_k mg, & \text{for} \quad v > 0 \end{cases}$$

- Viscous Friction



- All Frictions Combined



$$F_f(v) = \mathbf{F_c} + \sigma_v v + F_s(v)$$

Nonlinear Systems

Essential Nonlinear Phenomena

Nonlinear System Behavior [4]

We will deal with dynamical systems that are modeled by a finite number of coupled first-order ordinary differential equations

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
&\;\vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}
$$

where $\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t and u_1, u_2, …, u_p are specified input variables. We call the variables x_1, x_2, …, x_n the state variables. They represent the memory that the dynamical system has of its past.

Nonlinear System [4]

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}
$$

and rewrite the n first-order differential equations as one n-dimensional first-order vector differential equation

State-space Model

$$\dot{x} = f(t, x, u) \tag{1.1}$$

$$y = h(t, x, u) \tag{1.2}$$

Output

(1.2) is associated with (1.1), thereby defining a q-dimensional *output* vector y that comprises variables of particular interest in the analysis of the dynamical system, (e.g., variables that can be physically measured or variables that are required to behave in a specified manner).

Special cases [5]

$$\dot{x} = f(t, x, u) \qquad (1.1)$$

- An important special case of (1.1) is when the input u is identically zero. In this case the equation takes the form

$$\dot{x} = f(t, x, 0) = f(t, x) \qquad (1.3)$$

Unforced state equation

Notice that there is no difference between the unforced system with u=0 or any other function u(x,t) i.e. u is not an arbitrary variable. Substituting $\boxed{u = \gamma(t, x)}$ in (1.1) yields the unforced state equation.

Special cases [5]

- Another special case of (1.3) occurs when f does not depend explicitly on time t; that is

$$\dot{x} = f(x) \qquad (1.4)$$

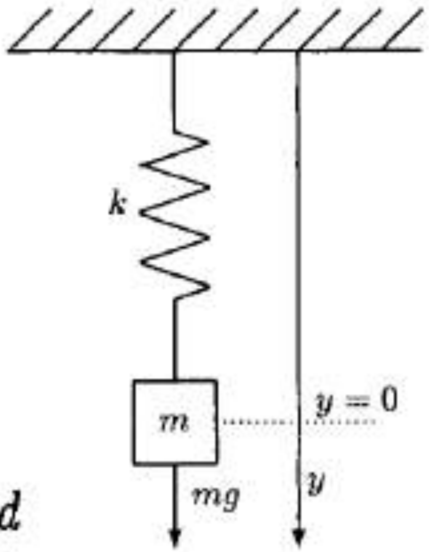
Autonomous or
Time Invariant

The behavior of an autonomous system is Invariant to SHIFTS in the time origin, since changing the time from t to $\tau = t - a$ does not change the right hand side of the state equation (1.4).

State Equation in Linear and Nonlinear Systems

Example: Linear System [5 ex.1.1]:

Consider the mass-spring system shown, using Newton's second law we obtain:



$$\begin{aligned} m\ddot{y} &= \sum forces \\ &= f(t) - f_k - f_\beta \end{aligned}$$

where y is the displacement from the reference position, f_β is the viscous friction force, and f_k represents the restoring force of the spring. Assuming linear properties, we have that $f_\beta = \beta\dot{y}$, and $f_k = ky$. Thus,

$$m\ddot{y} + \beta\dot{y} + ky = mg.$$

Defining states $x_1 = y, x_2 = \dot{y}$, we obtain the following state space realization

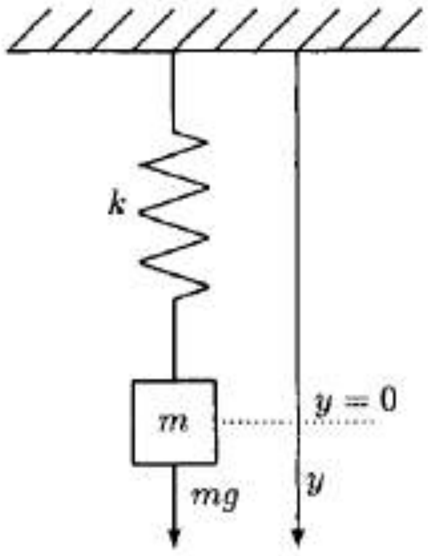
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{\beta}{m}x_2 + g \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$

Example:[5] continued

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g$$



If our interest is in the displacement y, then

$$y = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

thus, a state space realization for the mass–spring systems is given by

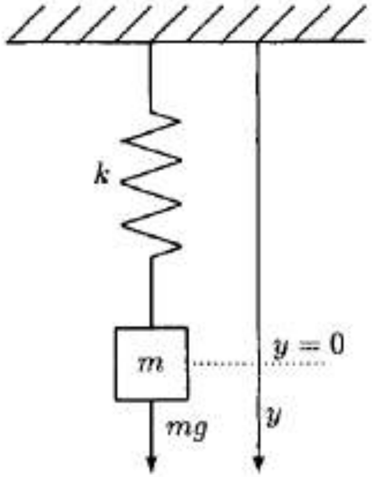
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\beta}{m} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = [0]$$

Example [5 ex. 1.2]: Nonlinear System

Consider again the spring-mass system



$$m\ddot{y} = \sum \textit{forces} = f(t) - f_k - f_\beta.$$

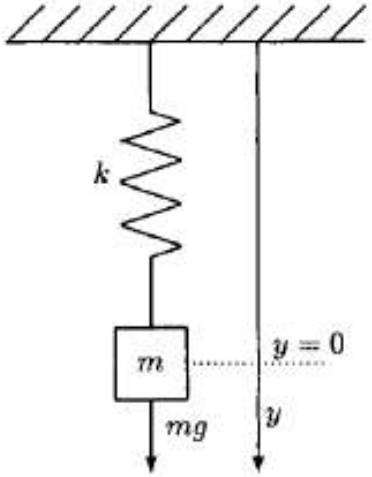
In Example 1.1 we assumed linear properties for the spring. We now consider the more realistic case of a hardening spring in which the force strengthens as y increases. We can approximate this model by taking

$$f_k = ky(1 + a^2y^2).$$

With this constant, the differential equation results in the following:

$$m\ddot{y} + \beta\dot{y} + ky + ka^2y^3 = f(t).$$

Example [5 ex. 1.2]: Nonlinear System



Defining state variables $x_1 = y, x_2 = \dot{y}$ *results in the following state space realization*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 - \frac{\beta}{m}x_2 + \frac{f(t)}{m} \end{cases}$$

which is of the form $\dot{x} = f(x, u)$. *In particular, if* $u = 0$, *then*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{k}{m}x_1 - \frac{k}{m}a^2x_1^3 - \frac{\beta}{m}x_2 \end{cases}$$

or $\dot{x} = f(x)$.

Equilibrium Point [5]

An important concept when dealing with the state equation is the concept of equilibrium point.

Definition 1.1 *A point $x = x_e$ in the state space is said to be an equilibrium point of the autonomous system*

$$\dot{x} = f(x)$$

if it has the property that whenever the state of the system starts at x_e, it remains at x_e for all future time.

According to the definition, the equilibrium points of the autonomous system in (1.4) are the real roots of the equation

$$\dot{x} = f(x) = 0$$

Equilibrium Point [5]

Example 1.3 *Consider the following first-order system*

$$\dot{x} = r + x^2$$

where r is a parameter. To find the equilibrium points of this system, we solve the equation $r + x^2 = 0$ and immediately obtain that

(i) *If $r < 0$, the system has two equilibrium points, namely $x = \pm\sqrt{r}$.*

(ii) *If $r = 0$, both of the equilibrium points in (i) collapse into one and the same, and the unique equilibrium point is $x = 0$.*

(iii) *Finally, if $r > 0$, then the system has no equilibrium points.* □

Essential Nonlinear Phenomena [4]

- *Multiple isolated equilibria.* A linear system can have only one isolated equilibrium point; thus, it can have only one steady-state operating point that attracts the state of the system irrespective of the initial state. A nonlinear system can have more than one isolated equilibrium point. The state may converge to one of several steady-state operating points, depending on the initial state of the system.

See Example 1.2 in page 7 in[1].

- *Finite escape time.* The state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state, however, can go to infinity in finite time.

See Fig. 1.3 of Example 1.2 in [1].

Equilibrium point [1]

Example 1.2 in page 7 in[1].

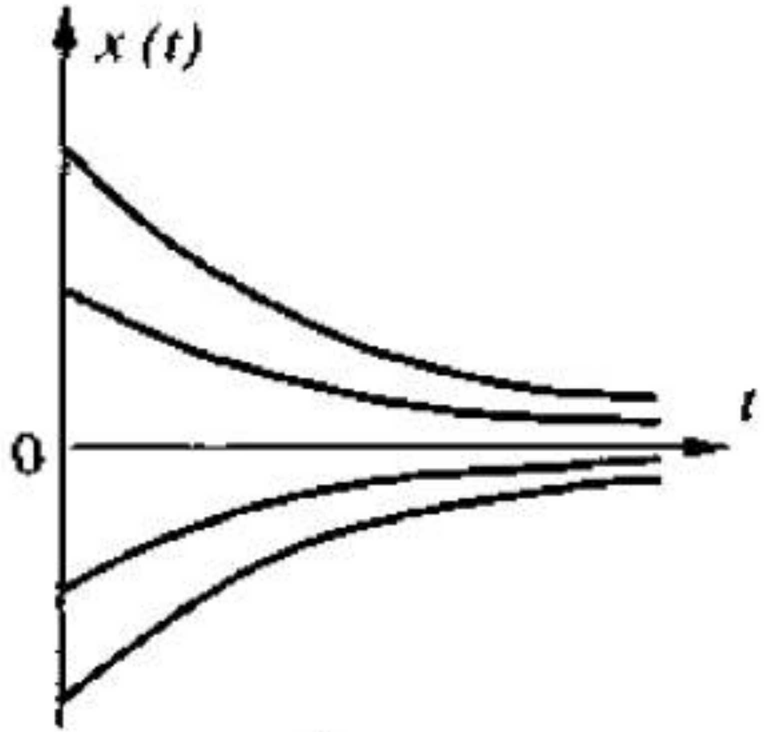
Consider the first order system

$$\dot{x} = -x + x^2$$

with initial condition $x(0) = x_0$. Its linearization is

$$\dot{x} = -x$$

The solution of this linear equation is $x(t) = x_0 e^{-t}$.



(a)

- *It is plotted in Figure 1.3(a) for various initial conditions.*
- *The linearized system clearly has a unique equilibrium point at x = 0.*

Equilibrium point [1]

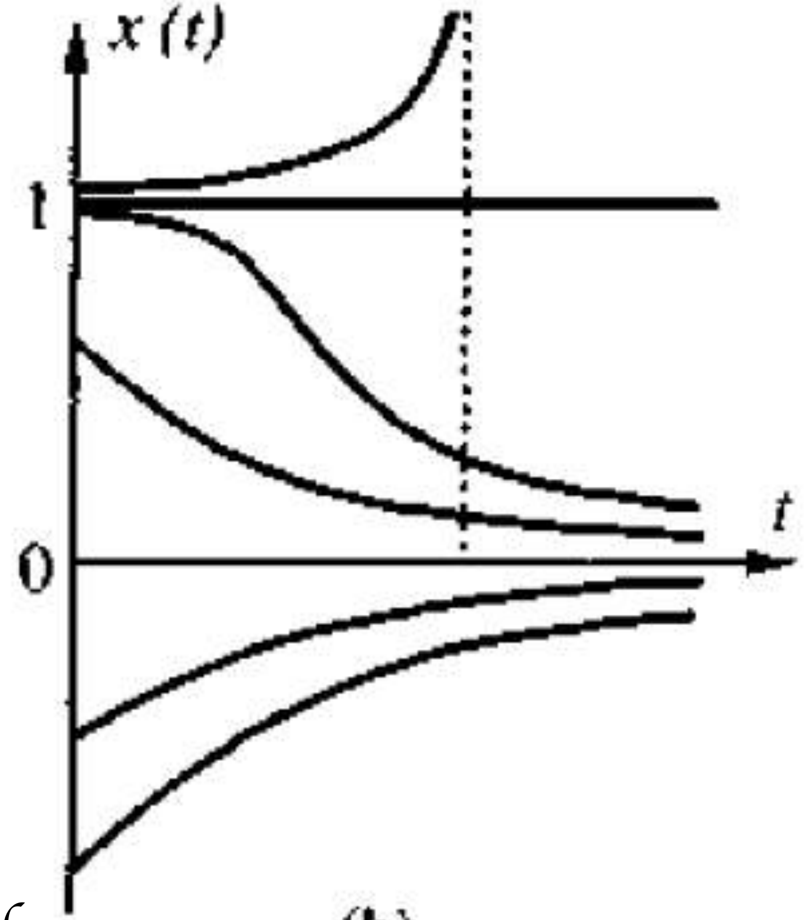
Example 1.2 in page 7 in[1].

- By contrast, for the nonlinear case $\dot{x} = -x + x^2$

- The system has two equilibrium points, $x_{e_1} = 1,\ x_{e_2} = 0$ and its qualitative behavior strongly depends on its initial condition.

  - $\dfrac{dx}{-x + x^2} = dt$, the actual response of the nonlinear dynamics $\dot{x} = -x + x^2$ can be found to be

$$x(t) = \frac{x_o e^{-t}}{1 - x_o + x_o e^{-t}}$$



(b)

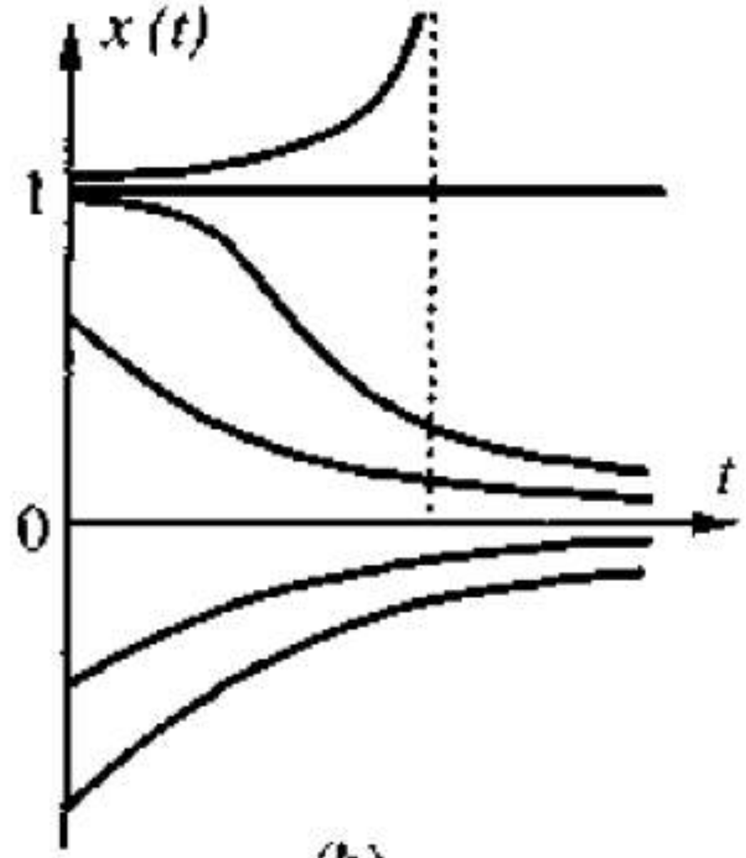
- This response is plotted in Figure 1.3(b) for various initial conditions.

Equilibrium point [1]

Example 1.2 in page 7 in[1].

The issue of motion stability can also be discussed with the aid of the above example.

- *For the linearized system:* stability is seen by noting that for any initial condition, the motion always converges to the equilibrium point x = 0.
- For the actual nonlinear system: motions starting with xo < 1 will indeed converge to the equilibrium point x = 0, those starting with xo> I will go to infinity (actually in finite time, a phenomenon known as finite escape time).
- This means that the stability of nonlinear systems may depend on initial conditions.



Essential Nonlinear Phenomena [4]

Limit Cycles

- Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation.
- These oscillations are called limit cycles, or self-excited oscillations.
- This important phenomenon can be simply illustrated by a famous oscillator dynamics, first studied in the 1920's by the Dutch electrical engineer Balthasar Van der Pol.

Essential Nonlinear Phenomena
Marginally stable systems and Limit Cycles [1, 6]

- Of course, *sustained oscillations can also be found in linear systems*, in the case of marginally stable linear systems (such as a mass-spring system without damping) or in the response to sinusoidal inputs.
- A linear system is marginally stable if its transient response neither decays nor grows, but remains constant or oscillates.
- A marginally–stable linear system has non–repeated poles on the imaginary axis and (possibly) poles in the left half plane.

Essential Nonlinear Phenomena

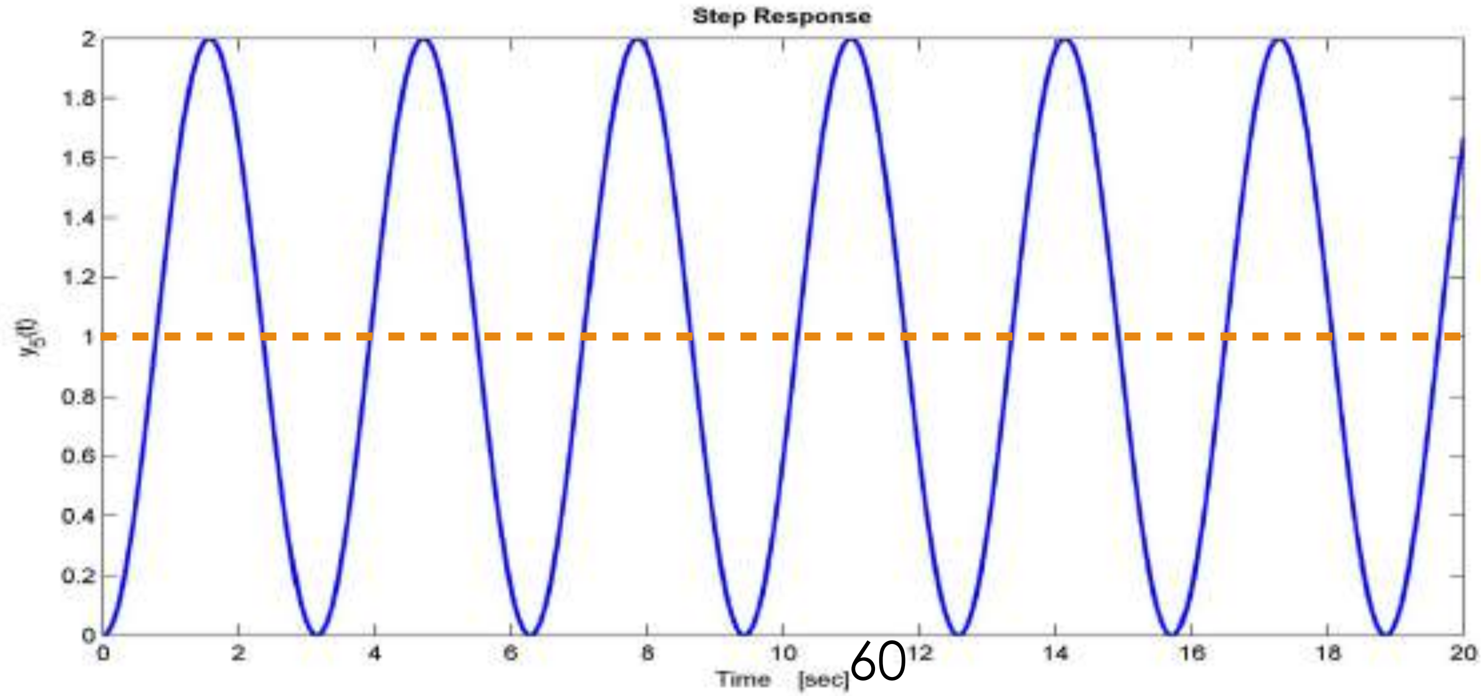
Marginally stable systems and Limit Cycles [6]

- Consider the following system (Transfer function): $$G(s) = \frac{4}{s^2 + 4}$$
- It has two imaginary poles $s_{1,2} = \pm 2i$
- If we take Step response for this undamped system we notice:
  - Transient response neither decays to zero, nor grows without bound.
  - It Oscillates indefinitely and the system is marginally stable

Essential Nonlinear Phenomena

Marginally stable systems and Limit Cycles [6]

- If we take Step response for this undamped system we notice: $G(s) = \dfrac{4}{s^2+4}$
  - Transient response neither decays to zero, nor grows without bound.
  - It Oscillates indefinitely and the system is marginally stable



H.W.
Simulate this system using 3 different initial conditions what do you observe?

Essential Nonlinear Phenomena

Marginally stable systems and Limit Cycles [4]

- *Limit cycles.* For a linear time-invariant system to oscillate, it must have a pair of eigenvalues on the imaginary axis, which is a nonrobust condition that is almost impossible to maintain in the presence of perturbations. Even if we do, the amplitude of oscillation will be dependent on the initial state. In real life, stable oscillation must be produced by nonlinear systems. There are nonlinear systems that can go into an oscillation of fixed amplitude and frequency, irrespective of the initial state. This type of oscillation is known as a limit cycle.

See Example 1.3 in page 8 in[1].

Essential Nonlinear Phenomena: Limit Cycles [1]

Example 1.3 [1]: Van der Pol Equation

The second-order nonlinear differential equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0$$

where m, c and k are positive constants, is the famous Van der Pol equation.

It can be regarded as describing a mass-spring-damper system with *a position-dependent damping coefficient* $2c(x^2 - 1)$.

Essential Nonlinear Phenomena: Limit Cycles [1]

Example 1.3 [1]: Van der Pol Equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0$$

- ☑ For large values of x the damping coefficient is positive and the damper removes energy from the system. This implies that the system motion has a *convergent tendency.*

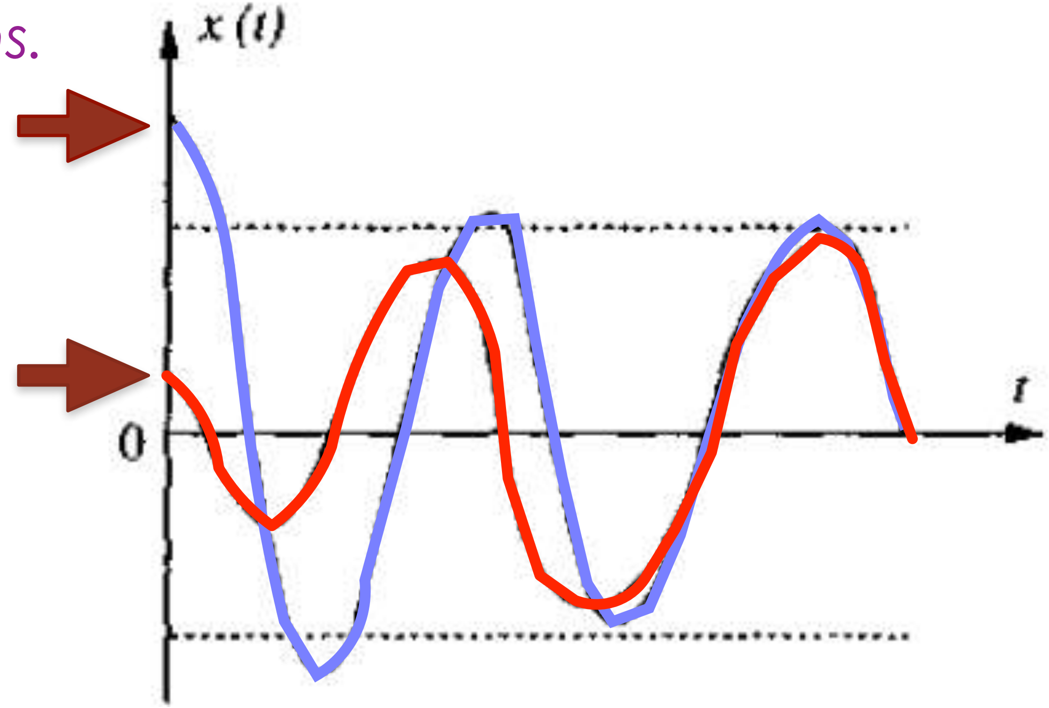
- ☑ For small values of x, the damping coefficient is negative and the damper adds energy into the system. This suggests that the system motion has *a divergent tendency.*

Essential Nonlinear Phenomena: Limit Cycles [1]

Example 1.3 [1]: Van der Pol Equation

$$m\ddot{x} + 2c(x^2 - 1)\dot{x} + kx = 0$$

- Therefore, because the nonlinear damping varies with x, the system motion can neither grow unboundedly nor decay to zero. Instead, it displays a sustained oscillation *independent of initial conditions.*
- This so-called *limit cycle* is sustained by periodically releasing energy into and absorbing energy from the environment, through the damping term.



Differences between
Linear Oscillations and Limit Cycles [1]

- Limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects.
  - ☑ **First,** the amplitude of the self-sustained excitation is independent of the initial condition, while the oscillation of a marginally stable linear system has its amplitude determined by its initial conditions.
  - ☑ **Second,** marginally stable linear systems are very sensitive to changes in system parameters (with a slight change capable of leading either to stable convergence or to instability), while limit cycles are not easily affected by parameter changes.

Limit Cycles [1]

- Limit cycles represent an important phenomenon in nonlinear systems.
- They can be found in many areas of engineering and nature.
- Aircraft wing fluttering, a limit cycle caused by the interaction of aerodynamic forces and structural vibrations, is frequently encountered and is sometimes dangerous.

  see https://www.youtube.com/watch?v=1DK-zGLK6GQ
- The hopping motion of a legged robot is another instance of a limit cycle. Limit cycles also occur in electrical circuits, e.g., in laboratory electronic oscillators.

Essential Nonlinear Phenomena [4]

- *Subharmonic, harmonic, or almost-periodic oscillations.* A stable linear system under a periodic input produces an output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, an example is the sum of periodic oscillations with frequencies that are not multiples of each other.

- *Chaos.* A nonlinear system can have a more complicated steady-state behavior that is not equilibrium, periodic oscillation, or almost-periodic oscillation. Such behavior is usually referred to as chaos. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.

A **deterministic system** is a system in which the current states of the system determine the future ones.

Chaos [1]

- By CHAOS we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is the unpredictability of the system output.
- Even if we have an exact model of a nonlinear system and an extremely accurate computer, the system's response in the long-run still cannot be well predicted.
- Chaos must be distinguished from random motion. In random motion, the system model or input contain uncertainty and, as a result, the time variation of the output cannot be predicted exactly (only statistical measures are available).
- In chaotic motion, the involved problem is deterministic, and there is little uncertainty in system model, input, or initial conditions.

Chaos [1]

- Chaos occurs mostly in strongly nonlinear systems. This implies that, for a given system, if the initial condition or the external input cause the system to operate in a highly nonlinear region, it increases the possibility of generating chaos.
- Chaos cannot occur in linear systems. Corresponding to a sinusoidal input of arbitrary magnitude, the linear system response is always a sinusoid of the same frequency.
- By contrast, the output of a given nonlinear system may display sinusoidal, periodic, or chaotic behaviors, depending on the initial condition and the input magnitude.

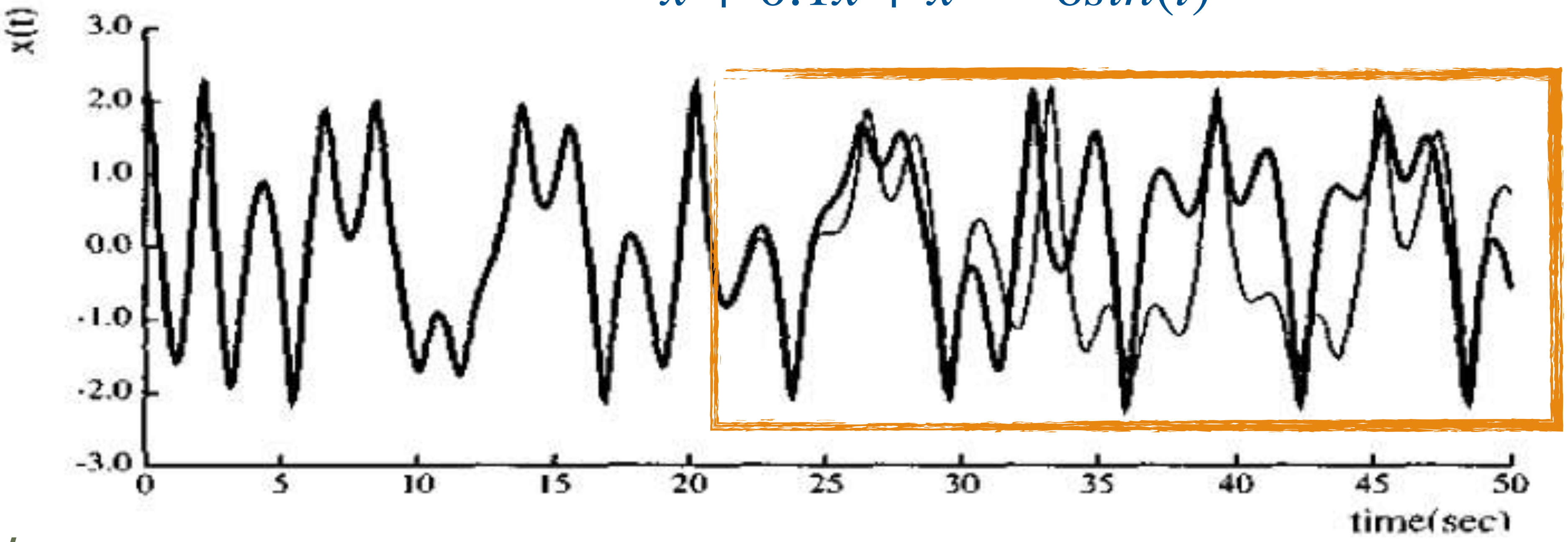
CHAOS Example [1]

- As an example of chaotic behavior, let us consider the simple nonlinear system
- $$\ddot{x} + 0.1\dot{x} + x^5 = 6sin(t)$$
- which may represent a lightly-damped, sinusoidally forced mechanical structure undergoing large elastic deflections.
- The following figure shows the responses of the system corresponding to <u>two almost identical initial conditions</u>, namely $x(0) = 2, \dot{x}(0) = 3$ (thick line) and $x(0) = 2.01, \dot{x}(0) = 3.01$ (thin line).

CHAOS Example [1]

- In the figure $x(0) = 2, \dot{x}(0) = 3$ (thick line) and $x(0) = 2.01, \dot{x}(0) = 3.01$ (thin line).

$$\ddot{x} + 0.1\dot{x} + x^5 = 6sin(t)$$



- Due to the presence of the strong nonlinearity in x^5, the two responses are radically different after some time.

CHAOS Example [1]

- Chaotic phenomena can be observed in many physical systems.
- The most commonly seen physical problem is turbulence in fluid mechanics (such as the swirls of incense stick). Atmospheric dynamics also display clear chaotic behavior, thus making long-term weather prediction impossible.

See the example in https://www.youtube.com/

- In the context of feedback control, it is of course of interest to know when a nonlinear system will get into a chaotic mode (so as to avoid it) and, in case it does, how to recover from it. Such problems are the object of active research.

Essential Nonlinear Phenomena: Bifurcation [1]

- **Bifurcation:** As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does in linear systems) and so can the number of equilibrium points.
- Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values.

Essential Nonlinear Phenomena: Bifurcation [1]

- The phenomenon of bifurcation, i.e., **quantitative change of parameters leading to qualitative change of system properties,** is the topic of bifurcation theory.

- For instance, the smoke rising from an incense stick first accelerates upwards (because it is lighter than the ambient air), but beyond some critical velocity breaks into swirls.

Essential Nonlinear Phenomena: Bifurcation [1]

- For instance, the smoke rising from an incense stick first accelerates upwards (because it is lighter than the ambient air), but beyond some critical velocity breaks into swirls.

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