Nonlinear systems Analysis

LECTURE 7

4th GRADE-CONTROL AND SYSTEMS ENGINEERING

University of Technology-Baghdad

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Limit Cycles [1]

- Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles, or self-excited oscillations.

- This important phenomenon can be simply illustrated by a famous oscillator dynamics, first studied in the 1920's by the Dutch electrical engineer Balthasar Van der Pol.

- Limit cycles are unique features of nonlinear systems.

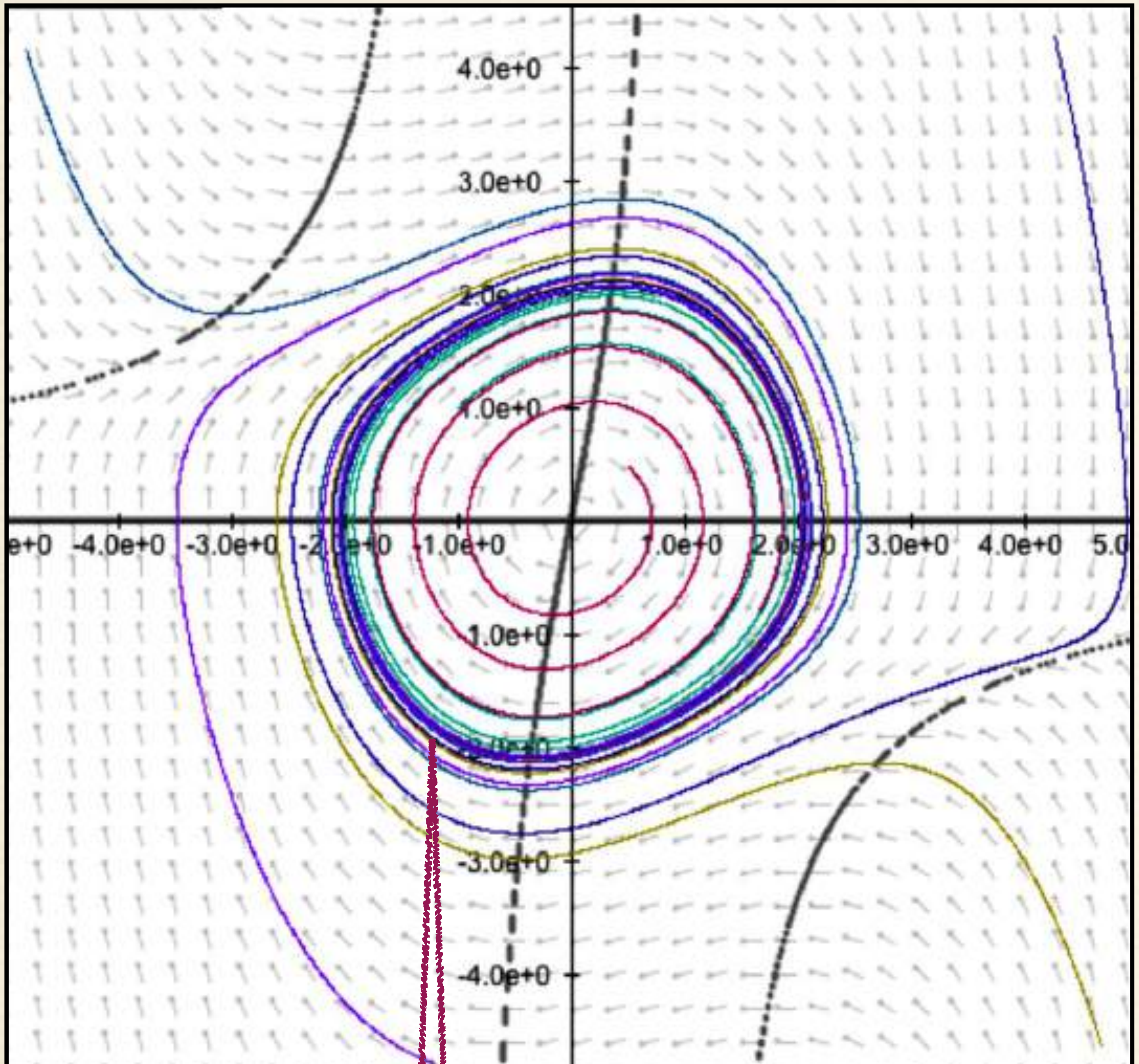
- In the phase plane, a limit cycle is defined as an *isolated closed curve*. The trajectory has to be both *closed*, indicating the periodic nature of the motion, and *isolated*, (isolated in the sense that neighboring trajectories are not closed, they are either converging or diverging from it)*.

* https://www.sciencedirect.com/topics/engineering/limit-cycle

LIMIT CYCLES [1]

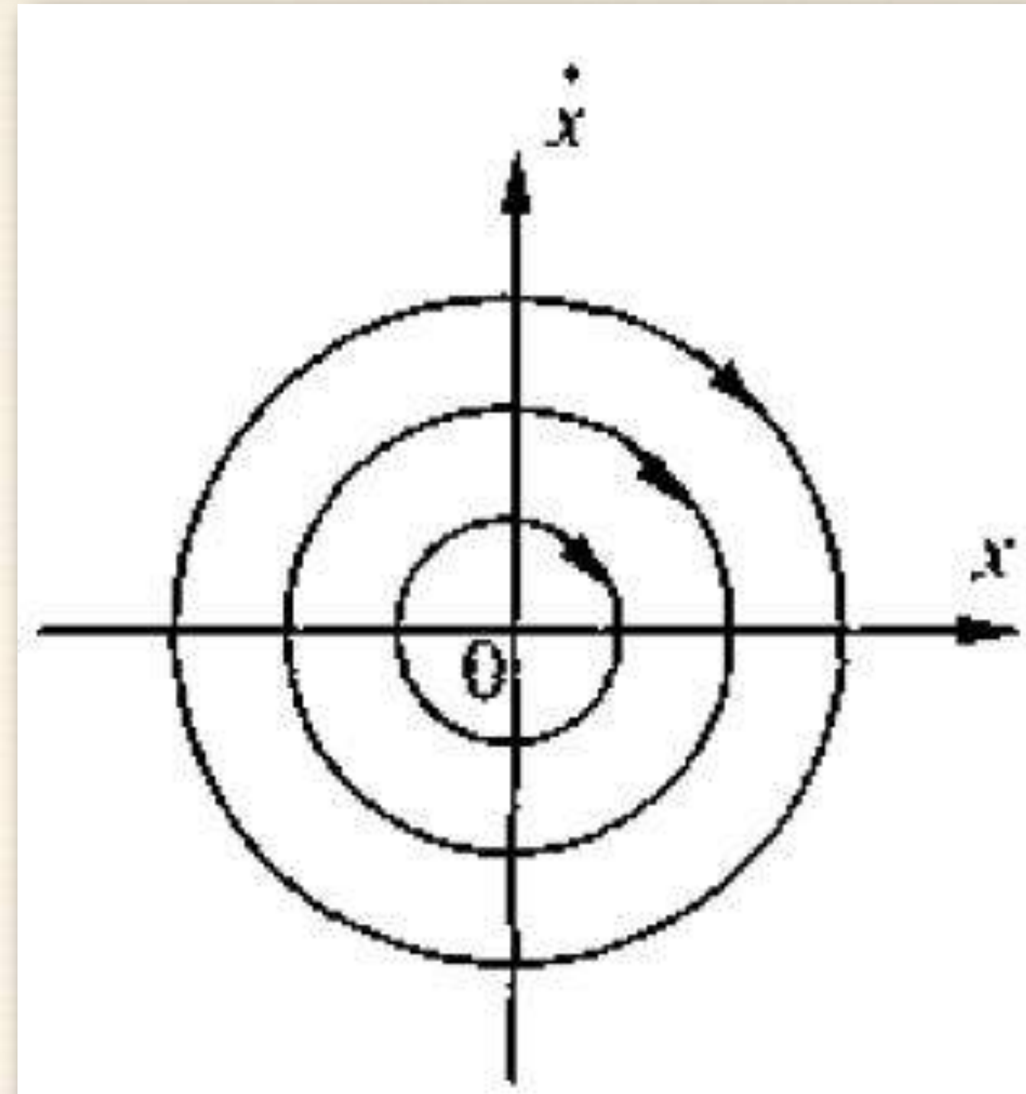
- A system oscillates when it has a nontrivial periodic solution $x(t + T) = x(t), \forall t \geq 0$, for some $T > 0$.
- The word "*nontrivial*" is used to exclude the constant solutions.
- We have already seen oscillation of linear system with eigenvalues $\pm j\beta$.
- The origin of the system is a center, and the trajectories are closed.
- Such oscillation where there is a continuum of closed orbits is referred to harmonic oscillator.

LIMIT CYCLE of Van Der Pol equation



Limit Cycle

Mass-Spring System



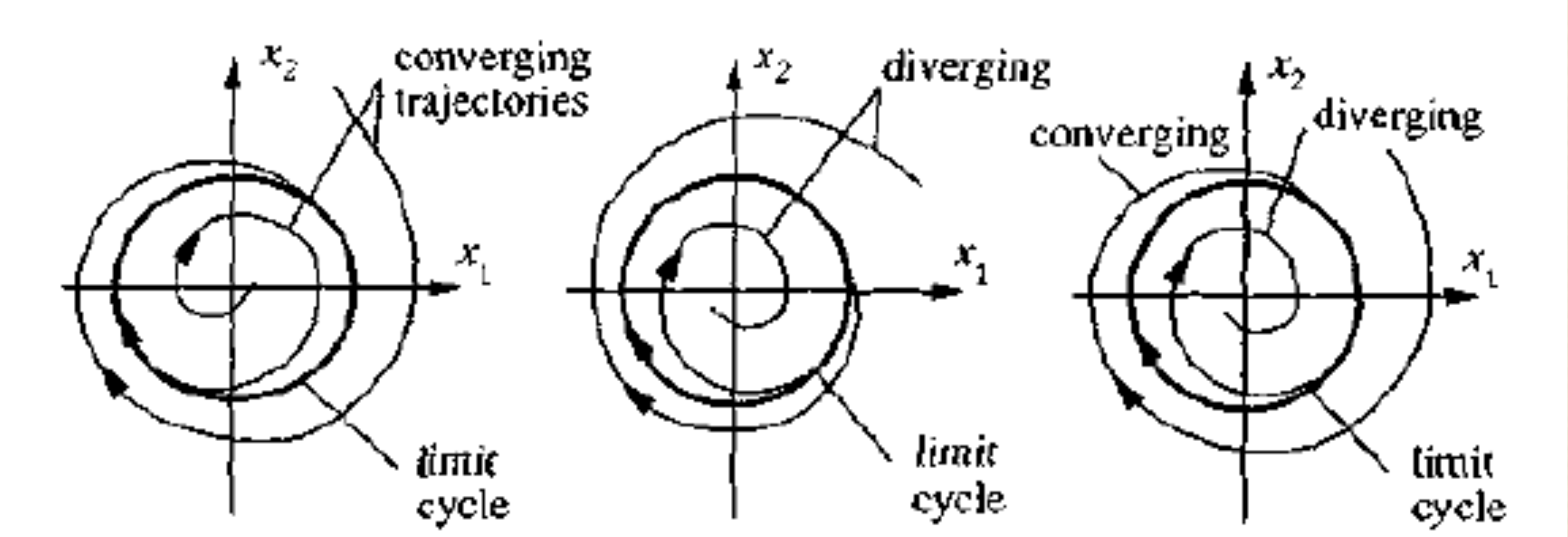
These are not considered limit cycles in this definition, because they <u>are not isolated.</u>

Kinds of LIMIT CYCLES [1]

1.Stable Limit Cycles: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$

2. Unstable Limit Cycles: all trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$

3.Semi-Stable Limit Cycles: some of the trajectories in the vicinity converge to it, while the others diverge from it as $t \to \infty$



Existence of LIMIT CYCLES

Periodic orbits in the plane are special in that they divide the the plane into a region inside the orbit and region outside it.

This makes it possible to obtain a criteria for detecting the existence of periodic orbits for second order systems which has no generalization for higher order systems.

The most celebrated criteria are the Poincaré, Poincaré- Bendixon and the Bendixon Theorems [1].

Poincaré-Bendixson Theorem [1] ***If a trajectory of the second-order autonomous system remains in a finite region Q, then one of the following is true:***

(a) the trajectory goes to an equilibrium point

(b) the trajectory tends to an asymptotically stable limit cycle

(c) the trajectory is itself a limit cycle.

This theorem is concerned with the asymptotic properties of the trajectories of second-order systems.

Existence of LIMIT CYCLES

Bendixson Theorem [1] ***For the nonlinear system***

$$\dot{x} = f(x)$$

,no limit cycle can exist in a region ***Q*** ***of the phase plane in which***

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

does not vanish and does not change sign.

This theorem provides a sufficient condition for the non-existence of limit cycles.

LIMIT CYCLES [1]

Example 2.8 of [1]: Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2 \qquad \dot{x}_2 = h(x_1) + 4x_1^2x_2$$

since

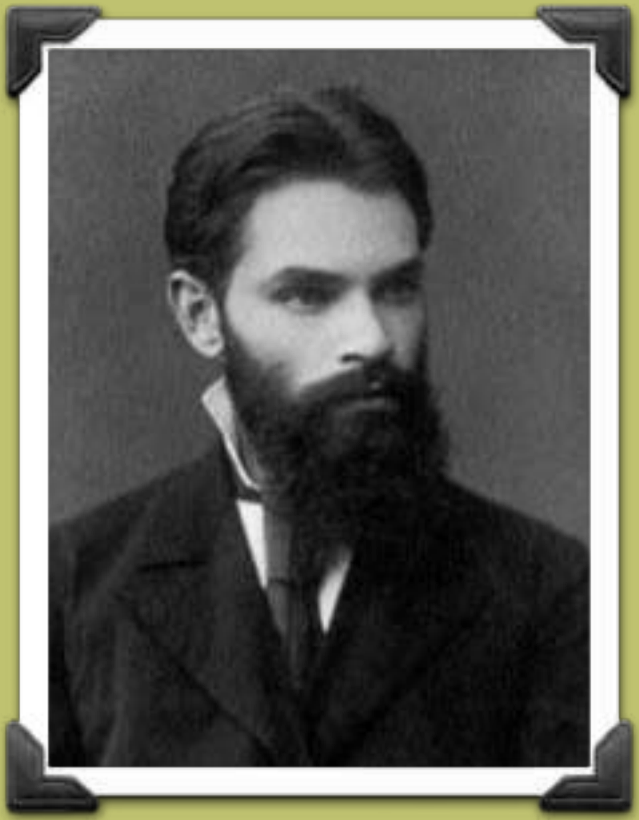
$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_2^2 + x_1^2)$$

which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane.

NONLINEAR SYSTEMS ANALYSIS

1. PHASE PLANE

2. LYAPUNOV THEORY



NONLINEAR SYSTEMS ANALYSIS

STABILITY

- Qualitatively, a system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after.

- The motions of a pendulum starting near its two equilibrium points, namely, the vertical up and down positions, are frequently used to illustrate unstable and stable behavior of a dynamic system.

- For aircraft control systems, a typical stability problem is intuitively related to the following question: will a trajectory perturbation due to a gust cause a significant deviation in the later flight trajectory?

  - Here, the desired operating point of the system is the flight trajectory in the absence of disturbance. Every control system, whether linear or nonlinear, involves a stability problem which should be carefully studied.

STABILITY DEFINITIONS

A few simplifying notations are defined at this point. Let **B_R** denote the spherical region (or ball) defined by || x || < R in state-space, and **S_R** the sphere itself, defined by || x || = R.

(Here $\|\cdot\|$ can be any vector norm.)

DEFINITION 1 (Equilibrium). A state x^* is an equilibrium point if $x(0) = x^*$ implies that $x(t) = x^*$ for all $t \geq 0$.

Any equilibrium x^* can be translated to the origin by redefining the state x as $x' = x - x^*$.

DEFINITION 2 The equilibrium state $x = 0$ is said to be <u>stable</u> if for any R>0, there exists r>0, such that if $\|x(0)\| < r$, then $\|x(t)\| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is <u>unstable</u>.

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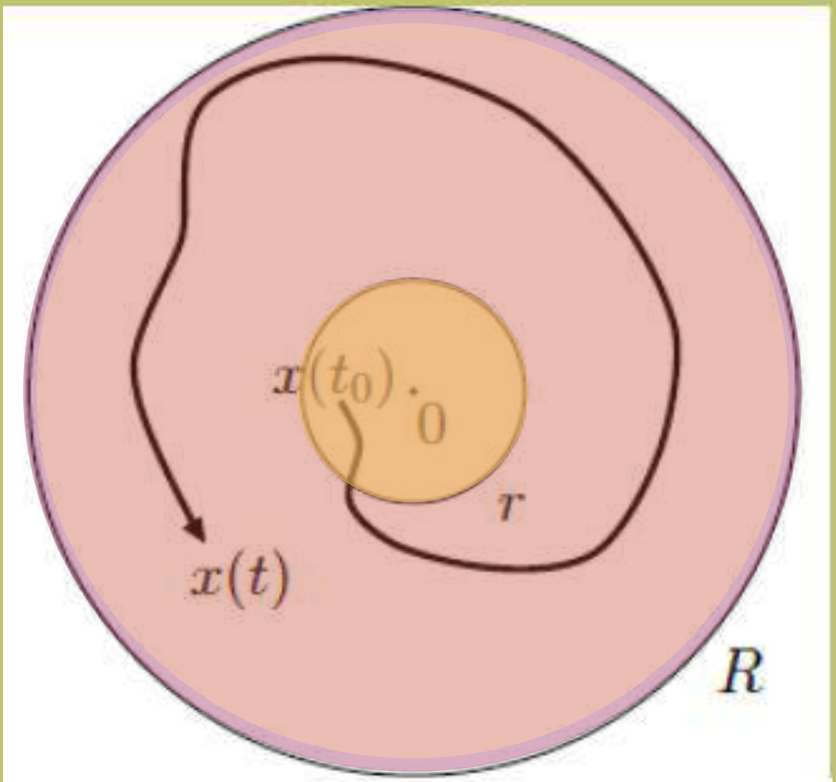
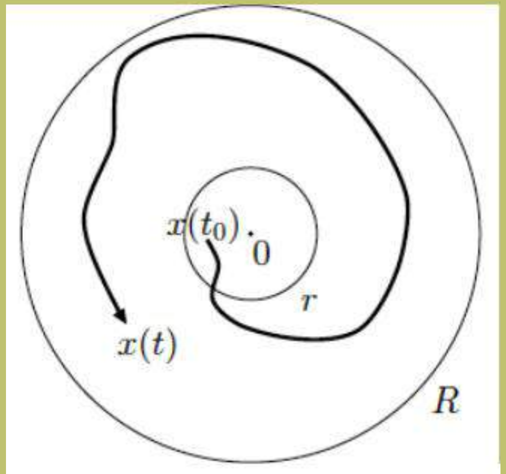


Stable equilibrium.

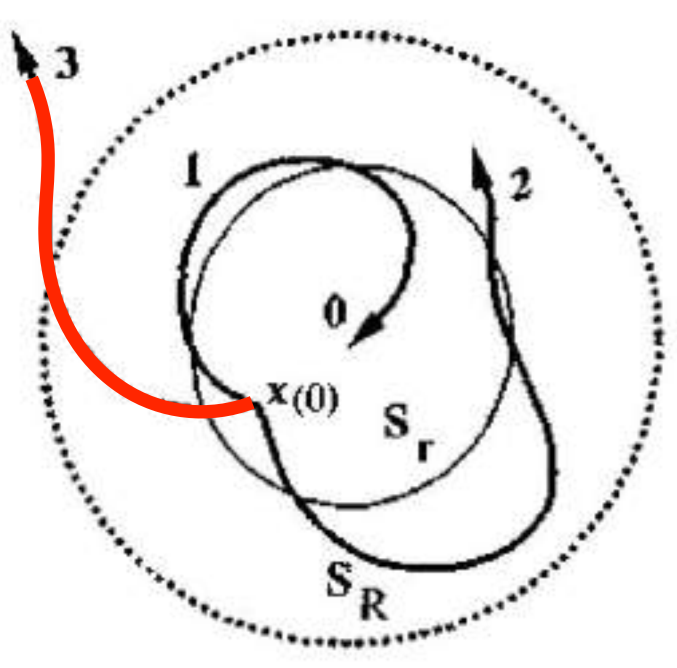
STABILITY DEFINITIONS



Stable equilibrium.

- Essentially, stability (also called stability in the sense of Lyapunov, or Lyapunov stability) means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it.

- an equilibrium point is unstable if there exists at least one ball B_R, such that for every r>0, no matter how small, it is always possible for the system trajectory to start somewhere within the ball **B_r** and eventually leave the ball **B_R**.
- Unstable nodes or saddle points in second-order systems are examples of unstable equilibria.
- Instability of an equilibrium point is typically undesirable, because it often leads the system into limit cycles or results in damage to the involved mechanical or electrical components.



STABILITY ANALYSIS [1]

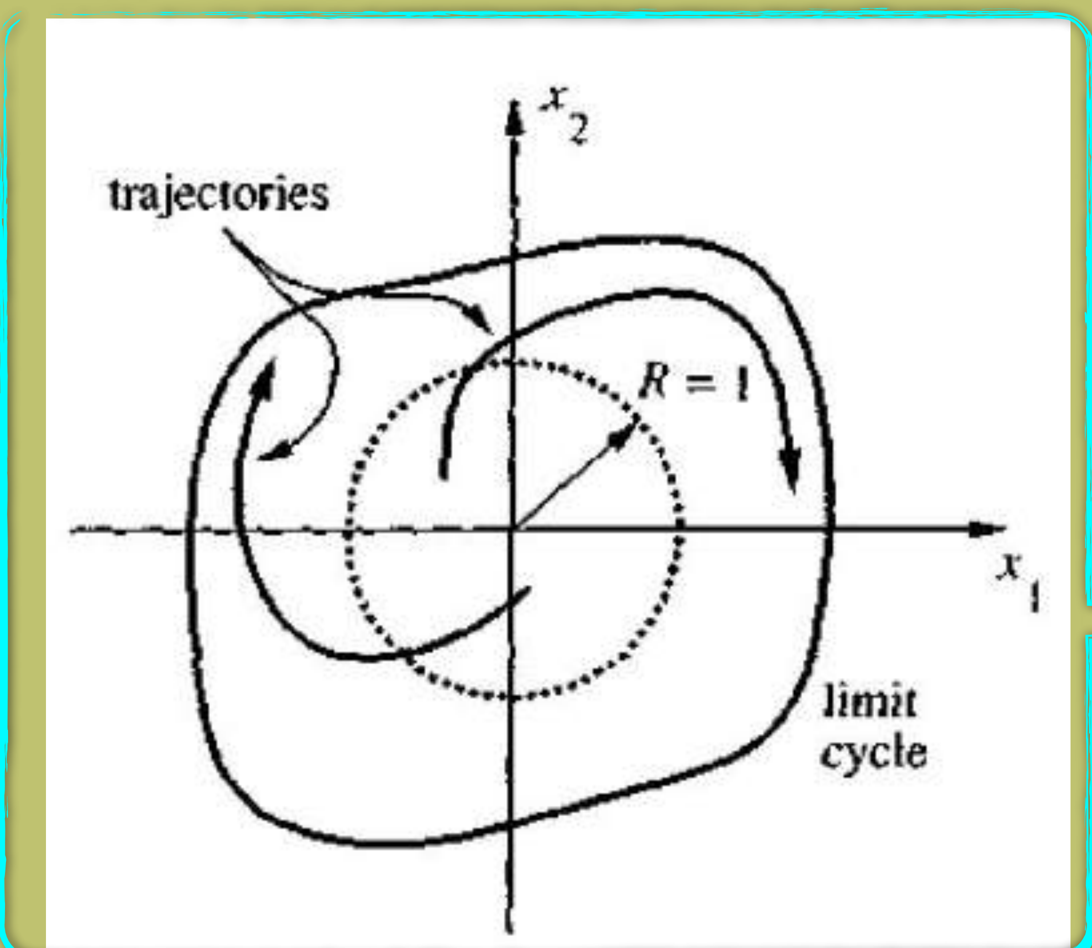
NOTE THAT:

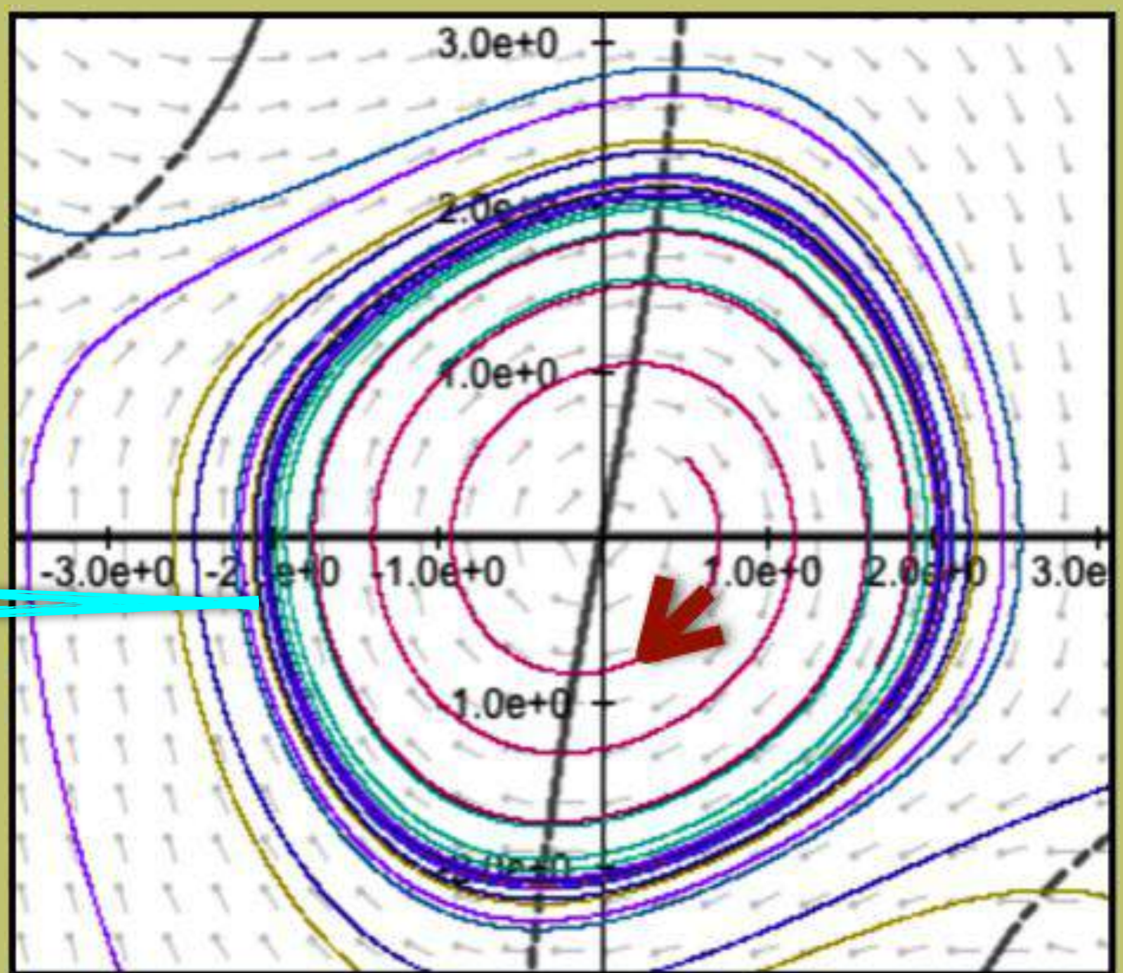
1. An equilibrium point $x^*=0$ may be unstable even though trajectories starting from points close to $x = 0$ do not tend to infinity.

2- The qualitative difference between instability and the intuitive notion of "blowing up" (all trajectories close to origin move further and further away to infinity). In linear systems, instability is equivalent to blowing up, because unstable poles always lead to exponential growth of the system states. However, for nonlinear systems, blowing up is only one way of instability.

STABILITY ANALYSIS [1]

This is the case for Van der Pol's equation, which has an unstable equilibrium at the origin. (All trajectories starting from points within the limit cycle eventually join the limit cycle, and therefore it is not possible to find r>0 in Definition 2 whenever R is small enough that some points on the closed curve of the limit cycle lie outside the set of points x satisfying $\|x\| < R.$)





CONCEPTS OF STABILITY [1]

- Since nonlinear systems may have much more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion.

- A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed.

- In many engineering applications, Lyapunov stability is not enough.

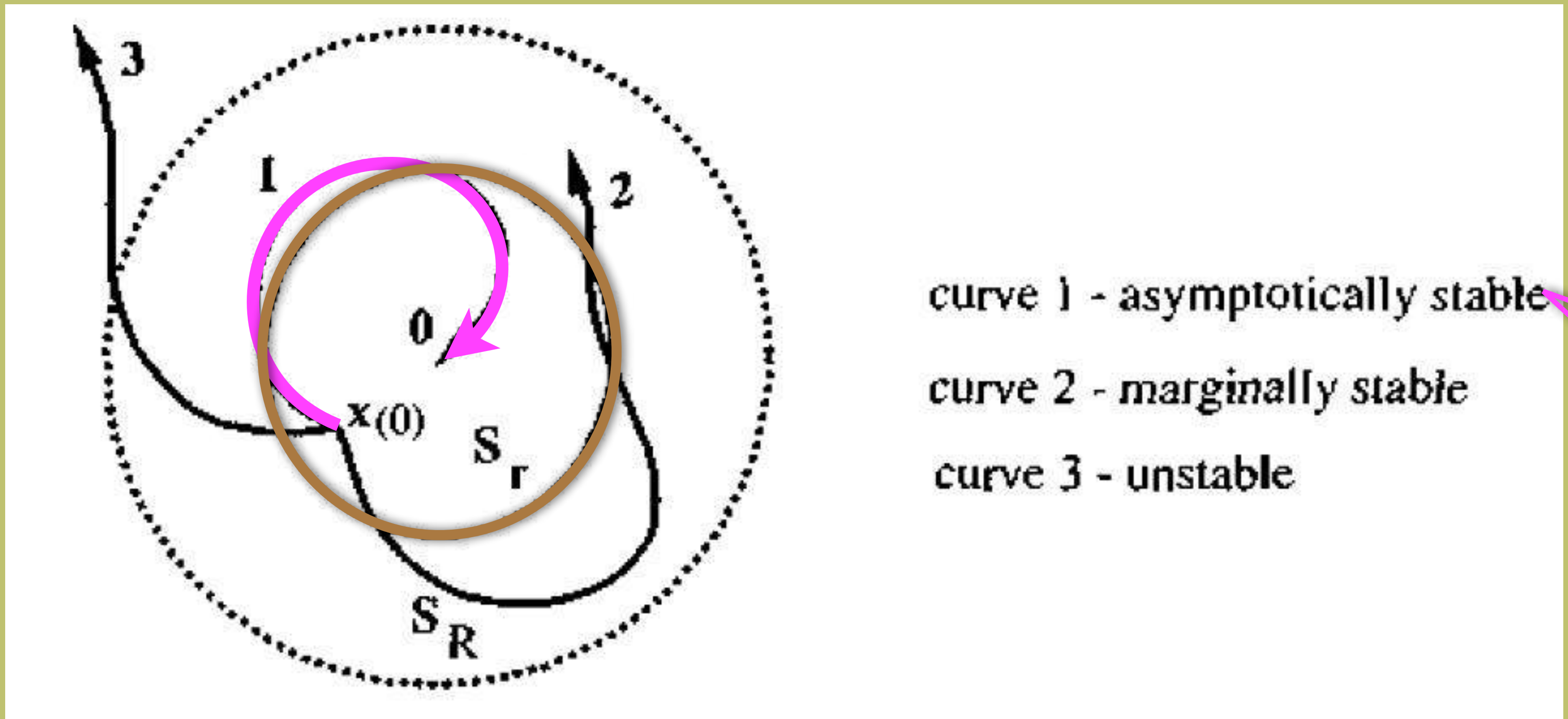
- For example, when a satellite's attitude is disturbed from its nominal position, we not only want the satellite to maintain its attitude in a range determined by the magnitude of the disturbance, i.e., Lyapunov stability, but also require that the attitude gradually go back to its original value. This type of engineering requirement is captured by the concept of *asymptotic stability.*

DEFINITION 3 *An equilibrium point 0 is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\| x(0) \| < r$ implies that $\|x(t)\| \to 0$ as $t \to \infty$.*

CONCEPTS OF STABILITY [1]

DEFINITION 3 *An equilibrium point 0 is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\| x(0) \| < r$ implies that $\|x(t)\| \to 0$ as $t \to \infty$.*

- Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to 0 actually converge to 0 as time t goes to infinity

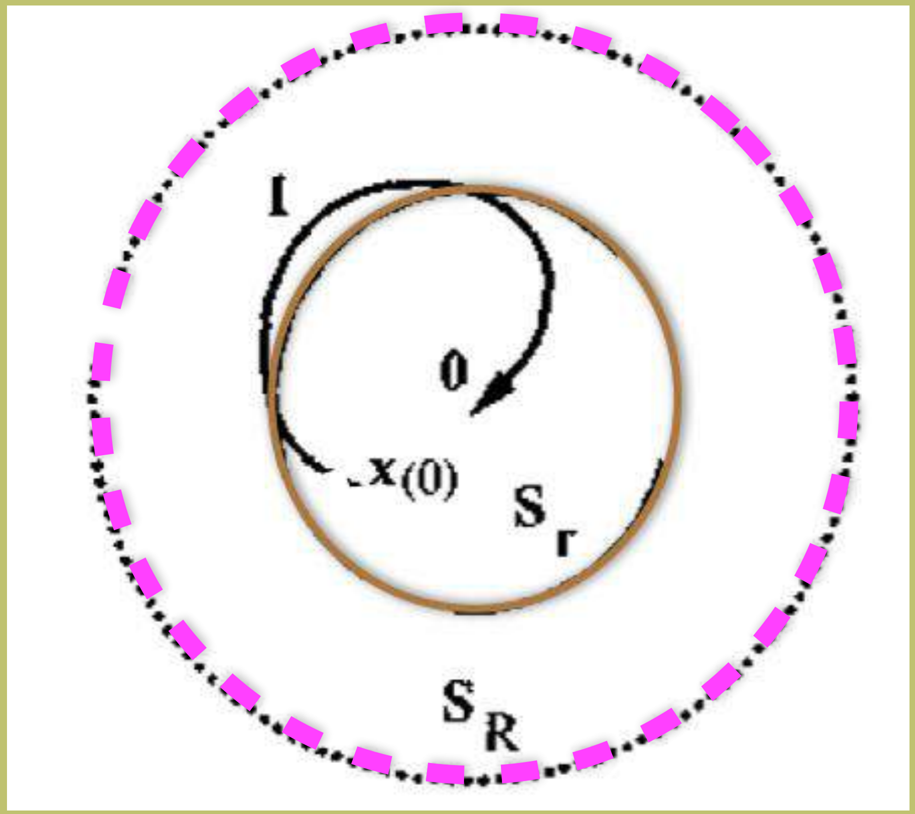


System trajectories starting from within the ball BR converge to the origin.

CONCEPTS OF STABILITY [1]

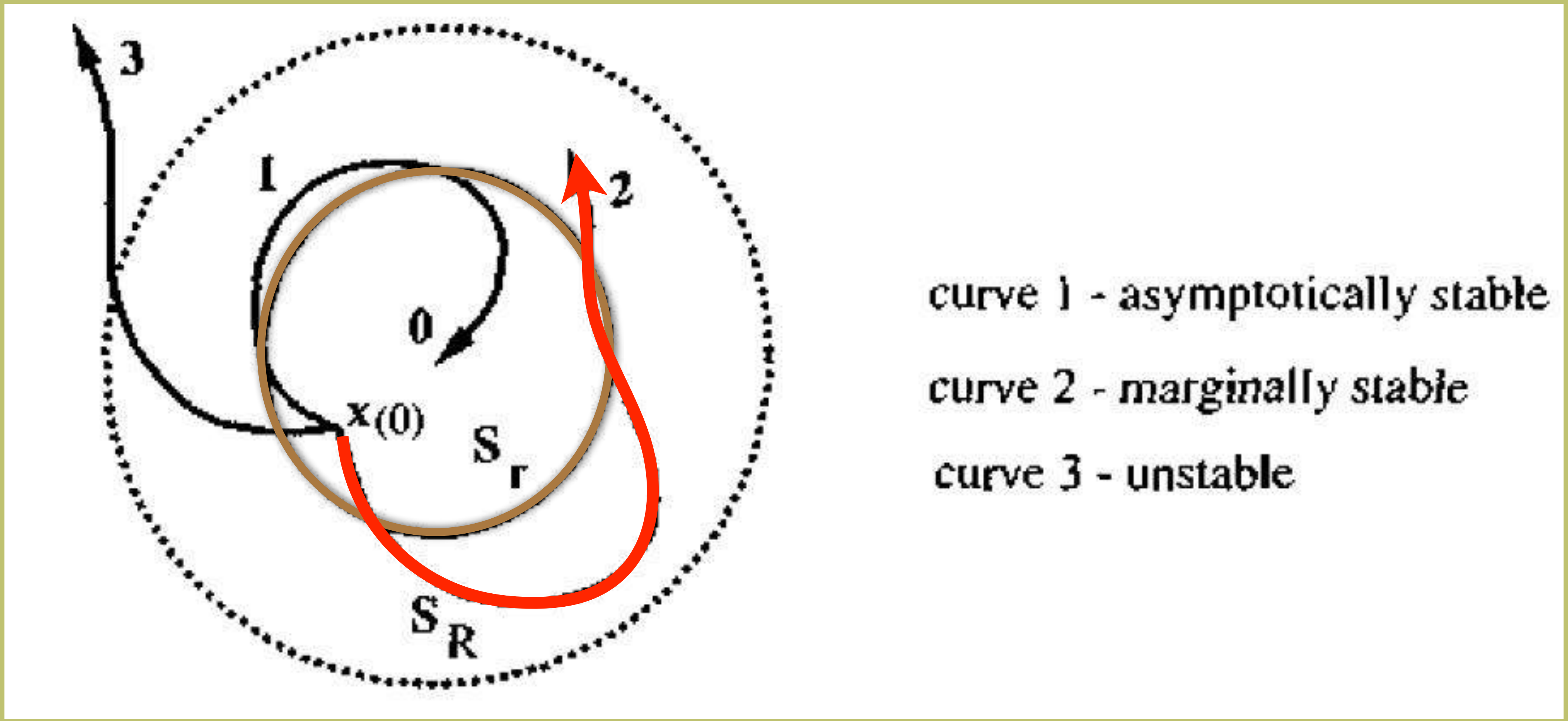
- The ball **B_R** is called *a domain of attraction* of the equilibrium point.

- The domain of attraction of the equilibrium point refers to the largest such region, i.e., to the set of all points such that trajectories initiated at these points eventually converge to the origin.



CONCEPTS OF STABILITY [1]

- An equilibrium point which is Lyapunov stable but not asymptotically stable is called *marginally stable*.

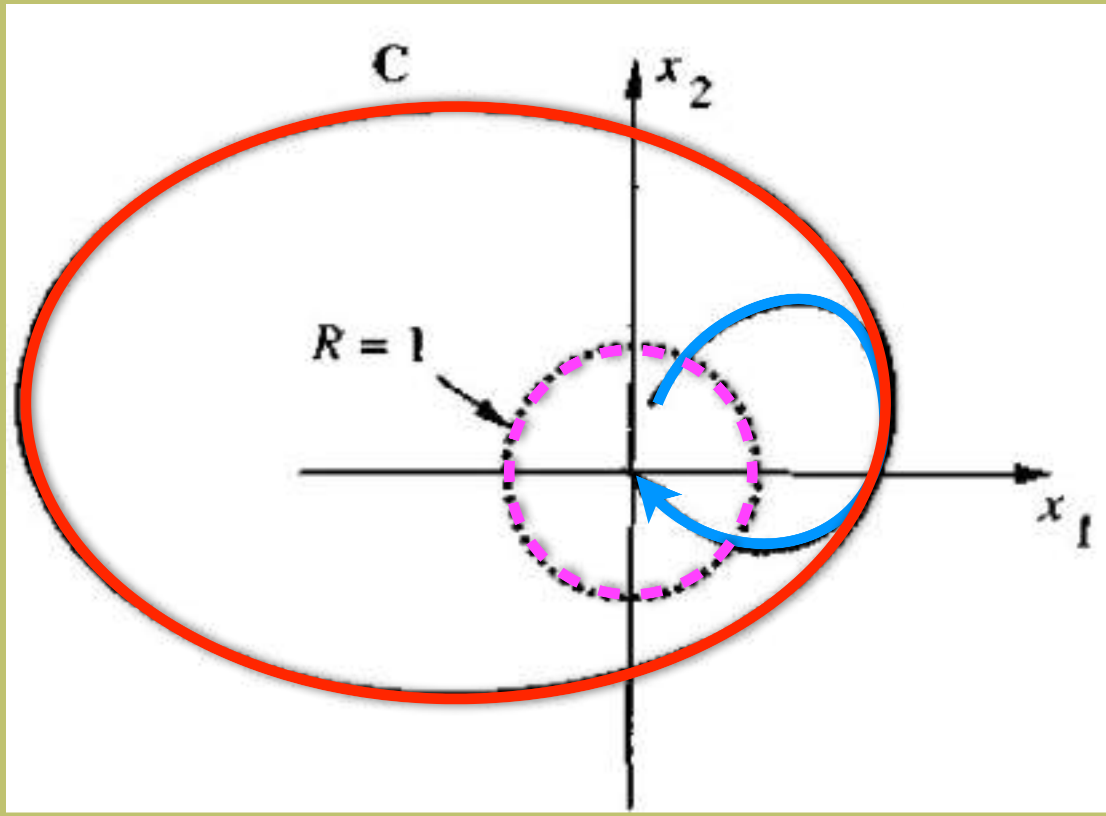


State convergence does not imply stability [1]

- It it easy to build counter-examples that show that state convergence does not necessarily imply stability.

  - For instance, a simple system studied by Vinograd has trajectories of the form shown in Figure below.

    All the trajectories starting from non-zero initial points within the unit disk first reach the curve C before converging to the origin. Thus, the origin is *unstable in the sense of Lyapunov*, despite the state convergence.



CONCEPTS OF STABILITY [1]

- In many engineering applications, it is still not sufficient to know that a system will converge to the equilibrium point after infinite time.

- There is a need to estimate *how fast* the system trajectory approaches the origin.

- The concept of *exponential stability* can be used for this purpose.

DEFINITION 4 *An equilibrium point 0 is exponentially stableif there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \| x(t) \| \leq \alpha \| x(0) \| e^{-\lambda t}$$

in some ball B_r *around the origin.*

CONCEPTS OF STABILITY [1]

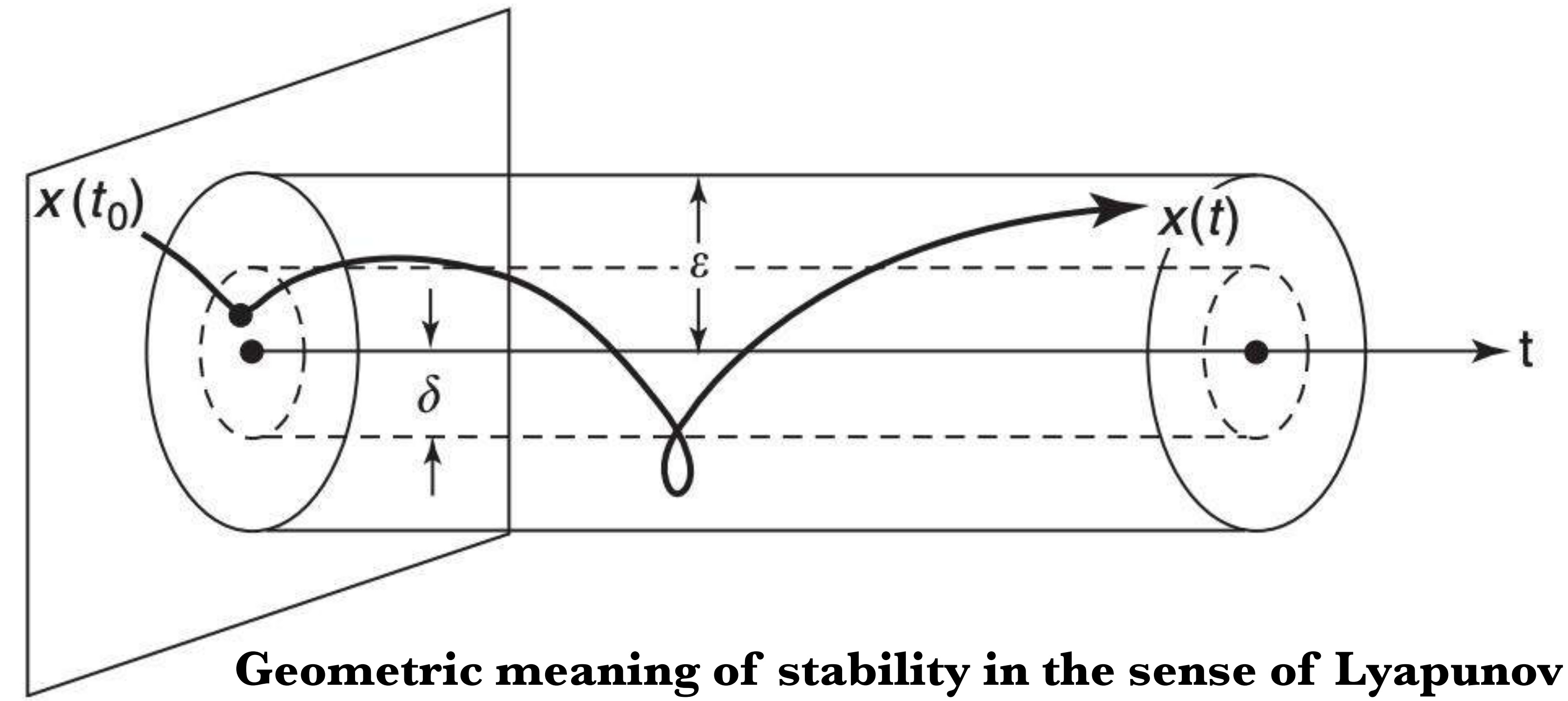
DEFINITION 4 *An equilibrium point 0 is exponentially stableif there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \| x(0) \| \leq r \Rightarrow \| x(t) \| \leq \alpha \| x(0) \| e^{-\lambda t}$$

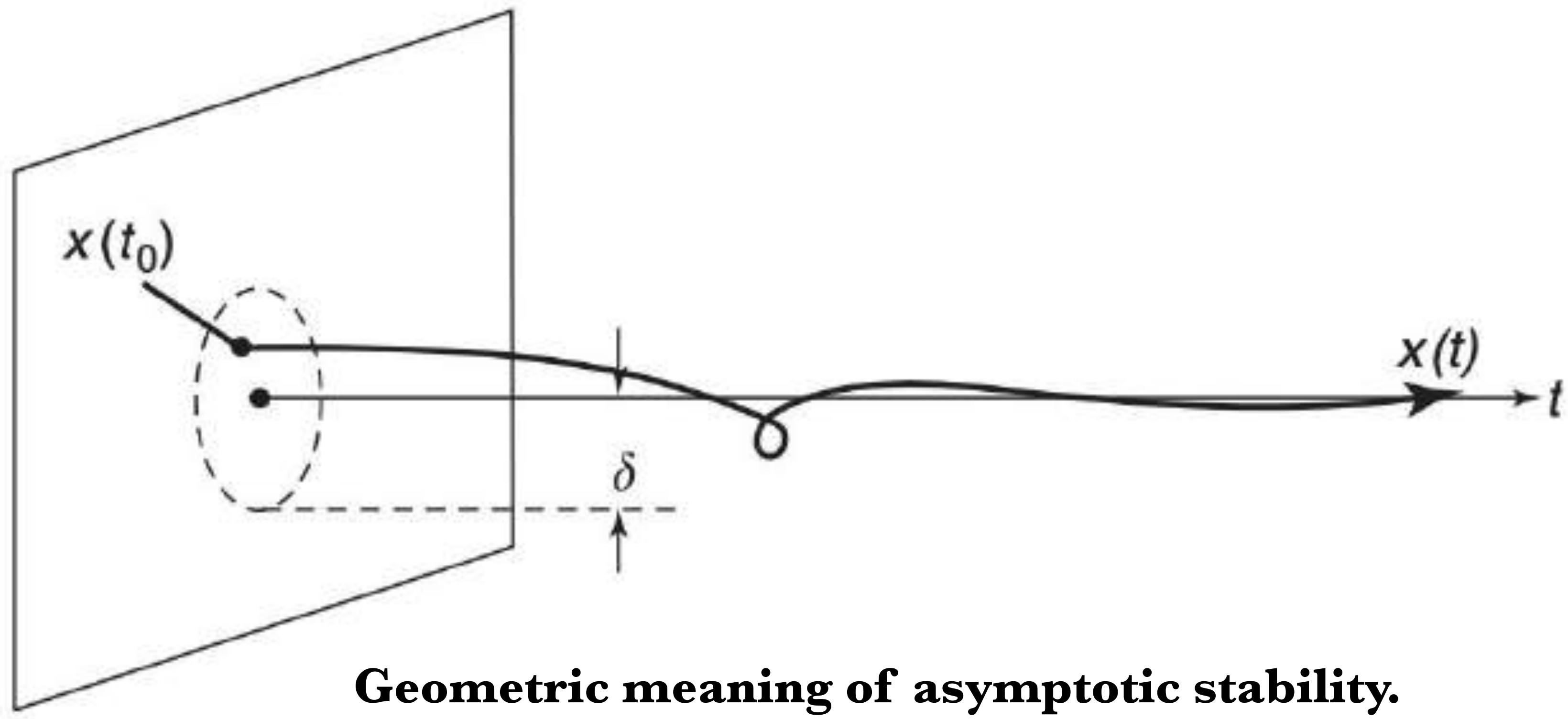
in some ball B_r *around the origin.*

- This means that the state vector of an exponentially stable system converges to the origin faster than an exponential function. The positive number λ is often called the rate of exponential convergence. (see examples in ref. [1]).

Lyapunov Stability [2]

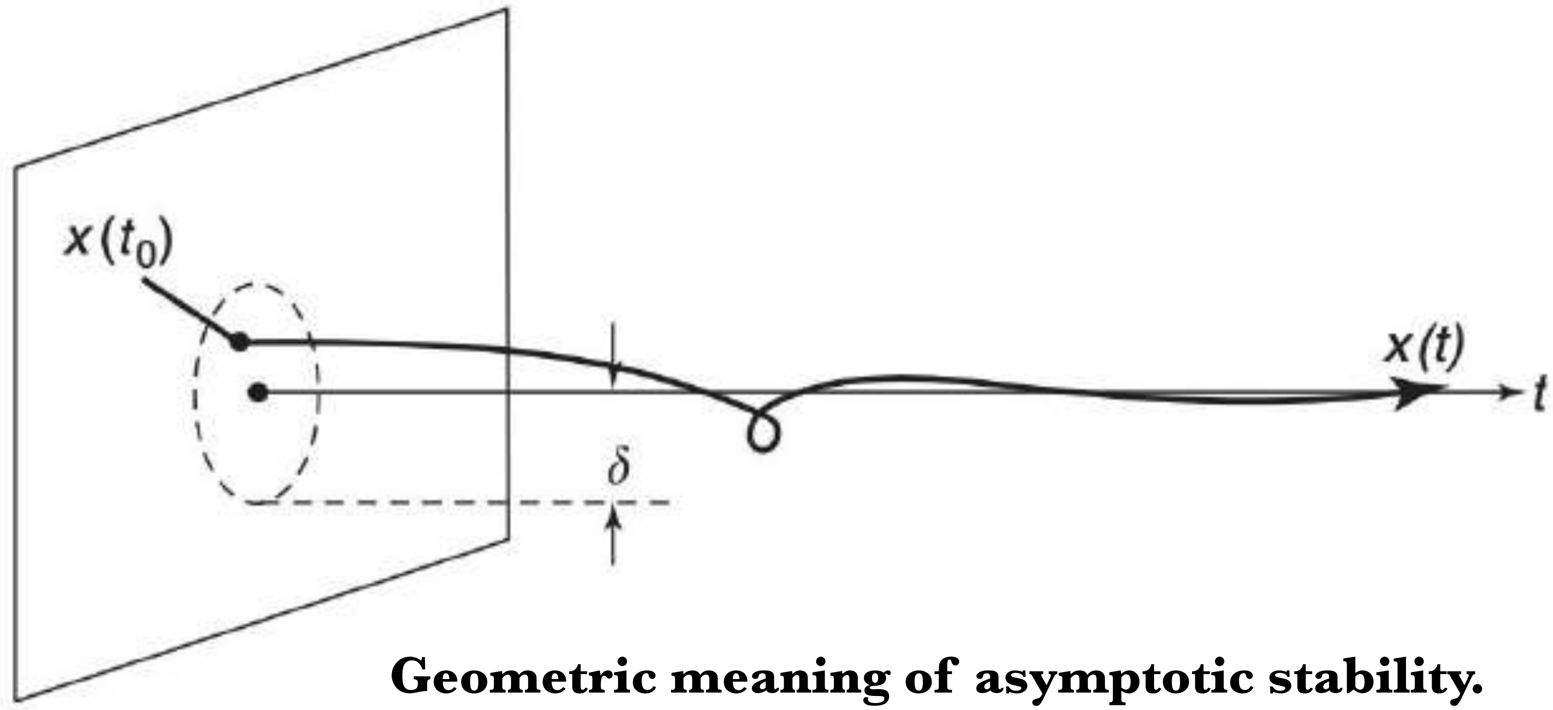


Geometric meaning of stability in the sense of Lyapunov

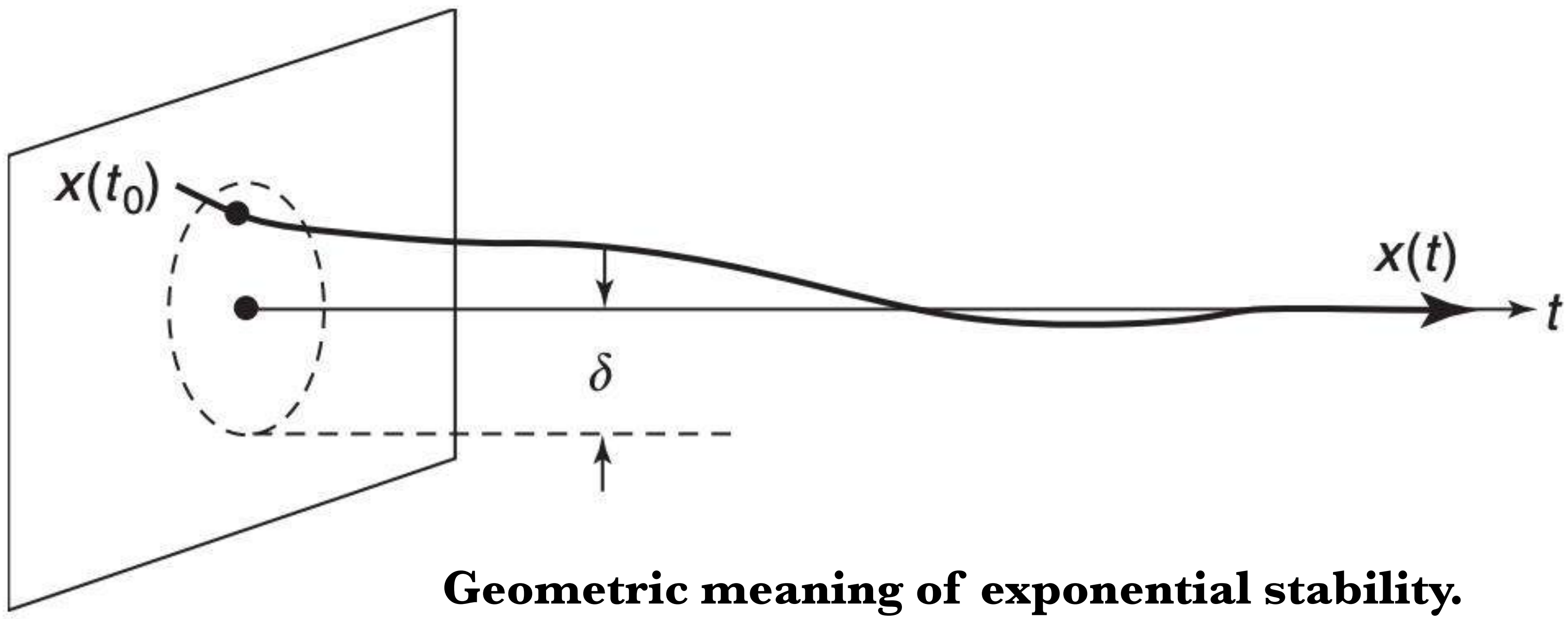


Geometric meaning of asymptotic stability.

Lyapunov Stability [2]



Geometric meaning of asymptotic stability.



Geometric meaning of exponential stability.

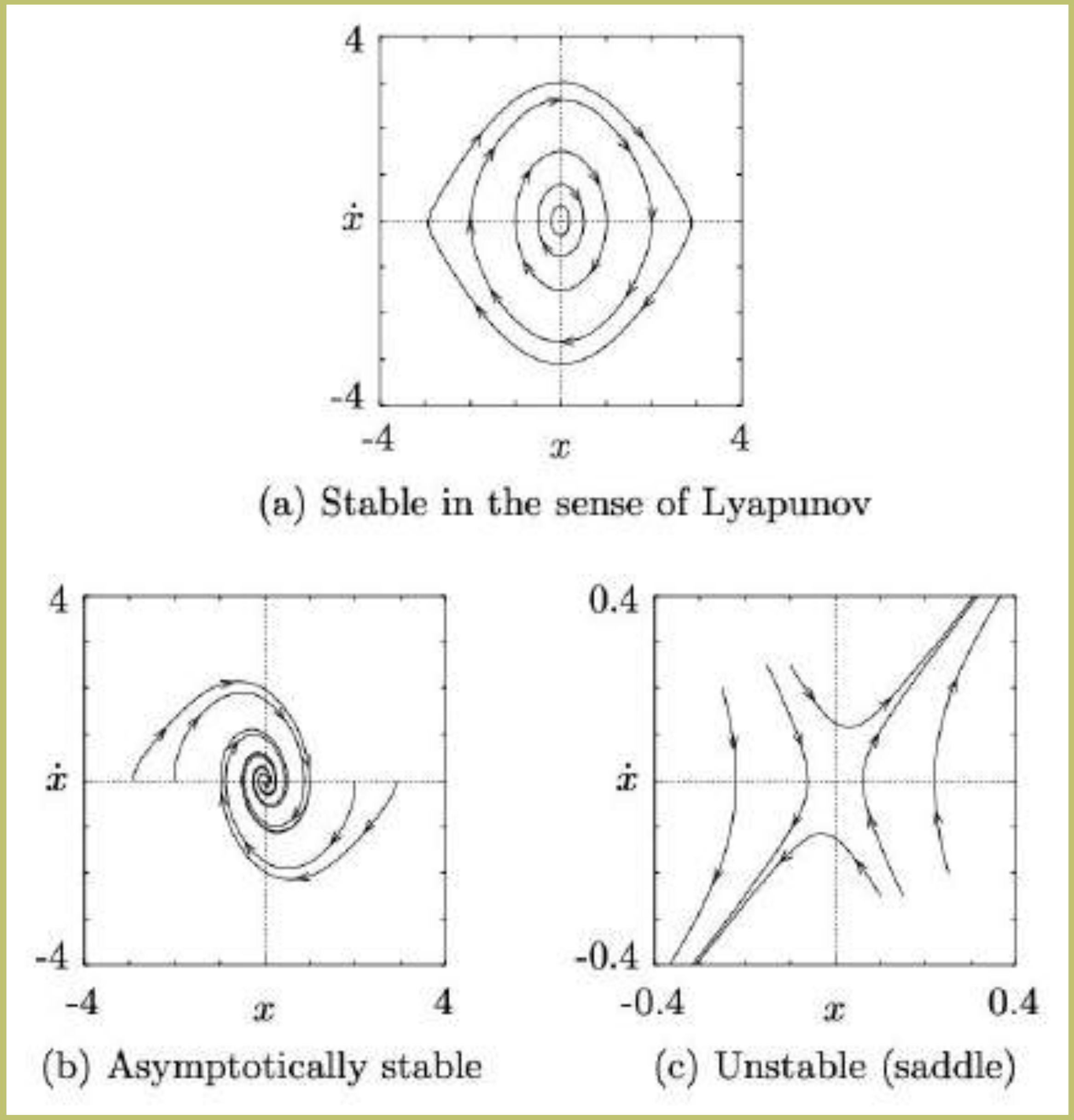
GLOBAL AND LOCAL STABILITY

Note that:

1. Asymptotic and exponential stability are local properties of a dynamic system since they only require that the state converges to zero from *a finite set of initial conditions* (known as a region of attraction): x where $\|x\| < r$.

2. If r can be taken to be infinite, then the system is respectively globally asymptotically stable or globally exponentially stable.

3. A strictly stable linear system is necessarily globally exponentially stable.



GLOBAL AND LOCAL STABILITY[1]

DEFINITION 5 *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.*

Note that:

1. Linear time-invariant systems are either asymptotically stable, or marginally stable, or unstable.

2. linear asymptotic stability is always global and exponential, and linear instability always implies exponential blow-up.

This explains why the refined notions of stability introduced here were not previously encountered in the study of linear systems. They are explicitly needed only for nonlinear systems.

Lyapunov Theory [1]

Basic Lyapunov theory comprises two methods introduced by Lyapunov, ***the indirect method and the direct method.***

The indirect method (linearization method) states that the stability properties of a nonlinear system in the close vicinity of an equilibrium point are essentially the same as those of its linearized approximation.

- This method serves as the theoretical justification for using linear control for physical systems, which are always inherently nonlinear.

The direct method is a powerful tool for nonlinear system analysis, and therefore the so-called Lyapunov analysis often actually refers to the direct method.

- The direct method is a generalization of the energy concepts associated with a mechanical system: *the motion of a mechanical system is stable if its total mechanical energy decreases all the time.*

LYAPUNOV THEORY
INDIRECT METHOD LINEARIZATION [3]

- What is linearization?

Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point.

- Why do we need it?

We have well stablished tools to analyze and stabilize linear systems.

LYAPUNOV THEORY
INDIRECT METHOD LINEARIZATION [3]

The METHOD

For the nonlinear *autonomous unforced* system

$$\dot{x} = f(x)$$

Let us use the constant matrix A to denote the Jacobian matrix of f with respect to x at x = 0

$$\dot{x} = Ax + g(x), A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

$g(x)$ contains the higher order terms (h.o.t.).

then the system $\dot{x} = Ax$ is called the linearization (or linear approximation) of the original nonlinear system at the equilibrium point 0.

Lyapunov Theory
Indirect Method Linearization [3]

Theorem1 (Lyapunov's linearization method)

* *If the linearized system is strictly stable (i.e, if all eigenvalues of A are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system).*

* *If the linearized system is unstable (i.e, if at least one eigenvalue of A is strictly in the right-half complex plane), then the equilibrium point is unstable (for the nonlinear system).*

* *If the linearized system is marginally stable (i.e, all eigenvalues of A are in the left-half complex plane, but at least one of them is on the /co axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable, or unstable for the nonlinear system).*

A summary of the theorem is that it is true by continuity. If the linearized system is strictly stable, or strictly unstable, then, since the approximation is valid "not too far" from the equilibrium, the nonlinear system itself is locally stable, or locally unstable. However, if the linearized system is marginally stable, the higher-order terms can have a decisive effect on whether the nonlinear system is stable or unstable

LYAPUNOV THEORY

Example 3.6: Consider the first order system

$$\dot{x} = a x + b x^5$$

The origin 0 is one of the two equilibrium points of this system. The linearization of this system around the origin is

$$\dot{x} = a x$$

The application of Lyapunov's linearization method indicates the following stability properties of the nonlinear system

- $a < 0$: asymptotically stable;
- $a > 0$: unstable;
- $a = 0$: cannot tell from linearization.

In the third case, the nonlinear system is

$$\dot{x} = b x^5$$

The linearization method fails while, as we shall see, the direct method to be described can easily solve this problem. □

Lyapunov Theory
Indirect Method Linearization [3]

Advantage:

Easy to apply

Disadvantages:

. If some eigenvalues of are zero, then we cannot draw any conclusion about stability of the nonlinear system.

. It is valid only if initial conditions are "*close*" to the equilibrium x*.

Lyapunov Theory
Indirect Method [1]

Lyapunov's linearization theorem shows that linear control design is a matter of consistency: one must design a controller such that the system remain in its “linear range".

- It also stresses major limitations of linear design: how large is the linear range?
- What is the extent of stability (how large is r in Definition 3) ?

These questions motivate a deeper approach to the nonlinear control problem, *Lyapunov’s direct method*.

LYAPUNOV THEORY
Direct method [1]

The DIRECT method is a powerful tool for nonlinear system analysis, and therefore the so-called Lyapunov analysis often actually refers to the direct method.

- The direct method is a generalization of the energy concepts associated with a mechanical system: ***the motion of a mechanical system is stable if its total mechanical energy decreases all the time.***

- In using the direct method to analyze the stability of a nonlinear system, the idea is to construct a scalar energy-like function (a Lyapunov function) for the system, and to see whether *it decreases*.

- The power of this method comes from its generality: it is applicable to all kinds of control systems, be they time-varying or time-invariant, finite dimensional or infinite dimensional.

- The limitation of the method lies ins the fact that it is often difficult to find a Lyapunov function for a given system.

Lyapunov Theory
Direct Method [1]

One important application is the design of nonlinear controllers.

- The idea is to somehow formulate *a scalar positive function of the system states*, and then choose a control law to make this function *decrease*.

- A nonlinear control system thus designed will be guaranteed to be *stable*.

- Such a design approach has been used to solve many complex design problems e.g. in robotics and adaptive control.

Lyapunov Theory
Direct Method [1]

Advantages:

- answers stability of nonlinear systems without explicitly solving dynamic equations
- can easily handle time varying systems $\dot{x} = f(x,t)$
- can determine asymptotic stability as well as plain stability
- can determine the region of asymptotic stability or the domain of attraction of an equilibrium

Lyapunov Theory
Direct Method [1]

Disadvantages of Lyapunov based Approach:

- There is no systematic way of obtaining Lyapunov functions.
- Lyapunov stability criterion provides only ***sufficient*** condition for stability.

Lyapunov Theory
Direct Method [3]

Lyapunov's second or direct METHOD: Consider the nonlinear system

$$\dot{x} = f(x), x^{*} \textit{ is an equilibrium state}$$

x*=(0,0), Suppose that there exists a function, called Lyapunov function $V(x)$, with the following properties:

$V(x) = 0 \; for \; x = x^* \; and \; V(x) > 0 \; for \; all \; x \neq x^*$ ($V(x)$ is Positive definite).

1. If $\dot{V}(x) \leq 0 \; for \; all \; x$ ($\dot{V}(x)$ is Negative semi-definite) then x* is Stable.

2. If $\dot{V}(x) < 0 \; for \; all \; x \neq x^*$ ($\dot{V}(x)$ is Negative definite) then the origin x* is Asymptotically Stable.

Lyapunov Theory
Direct Method [5,1]

To see the difference between positive definite and positive semi-definite, suppose that $x \in \mathbb{R}^2$ and let

$$V_1(x) = x_1^2 \qquad V_2(x) = x_1^2 + x_2^2.$$

Both V_1 and V_2 are always non-negative. However, it is possible for V_1 to be zero even if $x \neq 0$. Specifically, if we set $x = (0, c)$ where $c \in \mathbb{R}$ is any non-zero number, then $V_1(x) = 0$. On the other hand, $V_2(x) = 0$ if and only if $x = (0, 0)$. Thus $V_1(x)$ is positive semi-definite and $V_2(x)$ is positive definite.

A function $V(x)$ is negative definite if $-V(x)$ is positive definite; $V(x)$ is positive semi-definite if $V(0) = 0$ and $V(x) \geq 0 \; for \; x \neq 0$; $V(x)$ is negative semi-definite if $-V(x)$ is positive semi-definite. The prefix "semi" is used to reflect the possibility of V being equal to zero for $x \neq 0$.

Lyapunov Theory
Direct Method [3]

$$
\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} \\
&= \frac{\partial V}{\partial x} f(x) \\
&= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + \cdots + \frac{\partial V}{\partial x_n} f_n
\end{aligned}
$$

Recall the advantages of direct method:

- answers stability of nonlinear systems without explicitly solving dynamic equations
- can determine asymptotic stability as well as plain stability
- can determine the region of asymptotic stability or the domain of attraction of an equilibrium

Lyapunov Theory
Direct Method [H. Khalil]

Summary

- Lyapunov' Theorem: The origin is <u>stable</u> if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semi-definite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

EXAMPLE-1 [4]: CONSIDER THE SYSTEM $\dot{x}_1 = -x_1 + 4x_2\ , \quad \dot{x}_2 = -x_1 - x_2^3$

- The only equilibrium point for this system is the origin $x^*=(0,0)$.

- To investigate the stability of the origin let's propose a quadratic Lyapunov function $V = x_1^2 + ax_2^2$ where a is a positive constant to be determined.

- It is clear that V is positive definite on the entire state space $\mathbb{R}^2$. The derivative of V along the trajectories of the system is given by

$$\dot{V} = 2x_1\dot{x}_1 + 2ax_2\dot{x}_2$$
$$\dot{V} = -2x_1^2 + (8-2a)x_1x_2 - 2ax_2^4$$

If we choose a=4 then we can eliminate the cross term x_1x_2, and the derivative of V becomes

$$\dot{V} = -2x_1^2 - 8x_2^4$$

which is clearly a negative definite function on the entire state space. Therefore we conclude that x^* is a globally asymptotically stable equilibrium point.

LYAPUNOV THEORY

EXAMPLE-2 [3]: CONSIDER THE SYSTEM

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_2 - x_1^3 \end{aligned}$$

- Linearize this system around its equilibrium point, the characteristic equation of the linearized system is $s(s+3) = 0.$

- The -3 eigenvalue corresponds to the damping term but notice the existence of a zero eigenvalue from the lack of a linear term in the spring restoring force. The linearized version of the system cannot recognize the existence of a nonlinear spring term and it fails to produce a non-zero characteristic root related to the restoring force.

Lyapunov Theory

Example-2 [3]: Continued..

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -3x_2 - x_1^3$$

- Let's look at Lyapunov based approach.
- with equilibrium x*=(0,0) Let's try for a Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

We can see that $V(x) > 0$ for all x_1, x_2.

The time derivative of V is

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= x_1^3 x_2 + x_2(-3x_2 - x_1^3) \\ &= -3x_2^2 \quad , \dot{V}(x) \leq 0\end{aligned}$$

Negative semi definite. It follows then that x* is stable.

LYAPUNOV THEORY

EXAMPLE-2 [3]: CONTINUED..

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_2 - x_1^3 \end{aligned}$$

The time derivative of V is

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\ &= x_1^3 x_2 + x_2(-3x_2 - x_1^3) \\ &= -3x_2^2 \qquad , \dot{V}(x) \leq 0 \end{aligned}$$

It follows then that x* is stable but not asymptotically stable because $\dot{V}(x)$ is negative semi-definite. It is not negative definite because $\dot{V}(x) = 0 \;\; for \;\; x_2 = 0$ irrespective of the value of x_1, that is $\dot{V}(x) = 0$ along all the x1 axis. Therefore we can only conclude that the origin x* is stable. [H. Khalil Page.119]

Lyapunov Theory

Example-3[3]: For the system

$$\dot{x}_1 = -x_2 + a x_1 x_2^2$$

$$\dot{x}_2 = x_1 - b x_1^2 x_2, \quad a, b \text{ are const.} \quad a \neq b$$

Check the stability of the system using Lyapunov methods

- Solution: Find the equilibrium of the system by solving following equations:

$$-x_2 + a x_1 x_2^2 = 0$$

$$x_1 - b x_1^2 x_2 = 0$$

- Multiply the first equation by x_1 the second by x_2 and add them to get

$$x_1^2 x_2^2 (a - b) = 0$$

EXAMPLE-3[3]:CONTINUED...

- Solution: hence the equilibrium point $x^* = (0,0)$.
- The linearized system is $$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
- The characteristic equation is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

- Since the characteristic roots are purely imaginary, we can not draw any conclusion on the stability of the nonlinear system.

LYAPUNOV THEORY

EXAMPLE-3[3]:CONTINUED SOLUTION:

- Since the characteristic roots are purely imaginary, we can not draw any conclusion on the stability of the nonlinear system.

- Now we resort to Lyapunov based approach. Choose the Lyapunov function V(x) to be the sum of the kinetic and potential energy of the linear system (this does not work always!):

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

- We see that for all $V(x) > 0 \ \forall x_1, x_2$. Then

$$\begin{aligned} \dot{V}(x) &= x_1(-x_2 + a x_1 x_2^2) + x_2(x_1 - b x_1^2 x_2) \\ &= -x_1 x_2 + a x_1^2 x_2^2 + x_1 x_2 - b x_1^2 x_2^2 \\ &= (a-b) x_1^2 x_2^2 \end{aligned}$$

LYAPUNOV THEORY

EXAMPLE-3 [3]:CONTINUED SOLUTION:

- We see that for all $V(x) > 0 \;\; \forall x_1, x_2$. Then

$$
\begin{aligned}
\dot{V}(x) &= x_1(-x_2 + a x_1 x_2^2) + x_2(x_1 - b x_1^2 x_2) \\
&= -x_1 x_2 + a x_1^2 x_2^2 + x_1 x_2 - b x_1^2 x_2^2 \\
&= (a-b) x_1^2 x_2^2
\end{aligned}
$$

- Therefore, we see that
  - if a⟨b the system is <u>stable</u> (why?).
  - if a⟩b the system is <u>unstable</u>.

<u>Exercise:</u> Test the stability around the origin for the system $\dot{x} = y, \;\; \dot{y} = -\sin x$ **taking Lyapunov candidate function** $V(x, y) = \frac{y^2}{2} + 1 - \cos x$. **Is it Asymptotically stable or stable or unstable?**

Ans. (Stable)

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