Nonlinear Systems Analysis

LECTURE 4

4th GRADE-CONTROL AND SYSTEMS ENGINEERING

University of Technology-Baghdad

T. MOHAMMADRIDHA

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Nonlinear Systems Analysis [1]

Nonlinear equations, unlike linear ones, cannot in general be solved analytically, and therefore a complete understanding of the behavior of a nonlinear system is very difficult.

-Powerful mathematical tools like Laplace and Fourier transforms do not apply to nonlinear systems.

As a result, there are no systematic tools for predicting the behavior of nonlinear systems, nor are there systematic procedures for designing nonlinear control systems.

Serious efforts have been made to develop appropriate theoretical tools for it. Many methods of nonlinear control system analysis have been proposed.

Nonlinear systems Analysis

Many methods of nonlinear control system analysis have been proposed:

1. Phase Plane Analysis
2. Lyapunov Theory
3. Describing Function

NONLINEAR SYSTEMS

PHASE PLANE ANALYSIS

FOR AUTONOMOUS SYSTEMS

Phase Plane Properties [1]

Phase plane analysis is *a graphical method* for studying second-order systems, which was introduced in the 19th century by mathematicians such as *Henri Poincaré.*

- **The basic idea of the method: is to generate, in the phase plane, motion trajectories corresponding to various initial conditions.**
- **The goal is to examine the qualitative features of the trajectories.**
- **Information concerning *stability* and other motion patterns of the system can be obtained.**

Phase Plane Properties [1]

I. A graphical method: it allows us to visualize what goes on in a nonlinear system starting from various initial conditions, *without having to solve the nonlinear equations analytically.*

II. It is not restricted to small or smooth nonlinearities, but applies equally well to strong nonlinearities and to "hard" nonlinearities.

III. Some practical control systems can indeed be adequately approximated *as second-order systems,* and the phase plane method can be used easily for their analysis.

Phase Plane Disadvantage

- **It is restricted to second-order (or first order) systems, because the graphical study of higher-order systems is computationally and geometrically complex.**

PHASE PLANE ANALYSIS

CONCEPTS

Concepts of Phase Plane Analysis

Phase Plane portrait: For the following second order autonomous system:

$$\dot{x}_1(t) = f_1(x_1, x_2) \qquad (1)$$

$$\dot{x}_2(t) = f_2(x_1, x_2) \qquad (2)$$

- ***x*(0)=*xo*.**

- **Geometrically, the state space of this system is a *plane* having x_1 and x_2 as coordinates.**

- **The plane is called *state* or *phase plane*.**

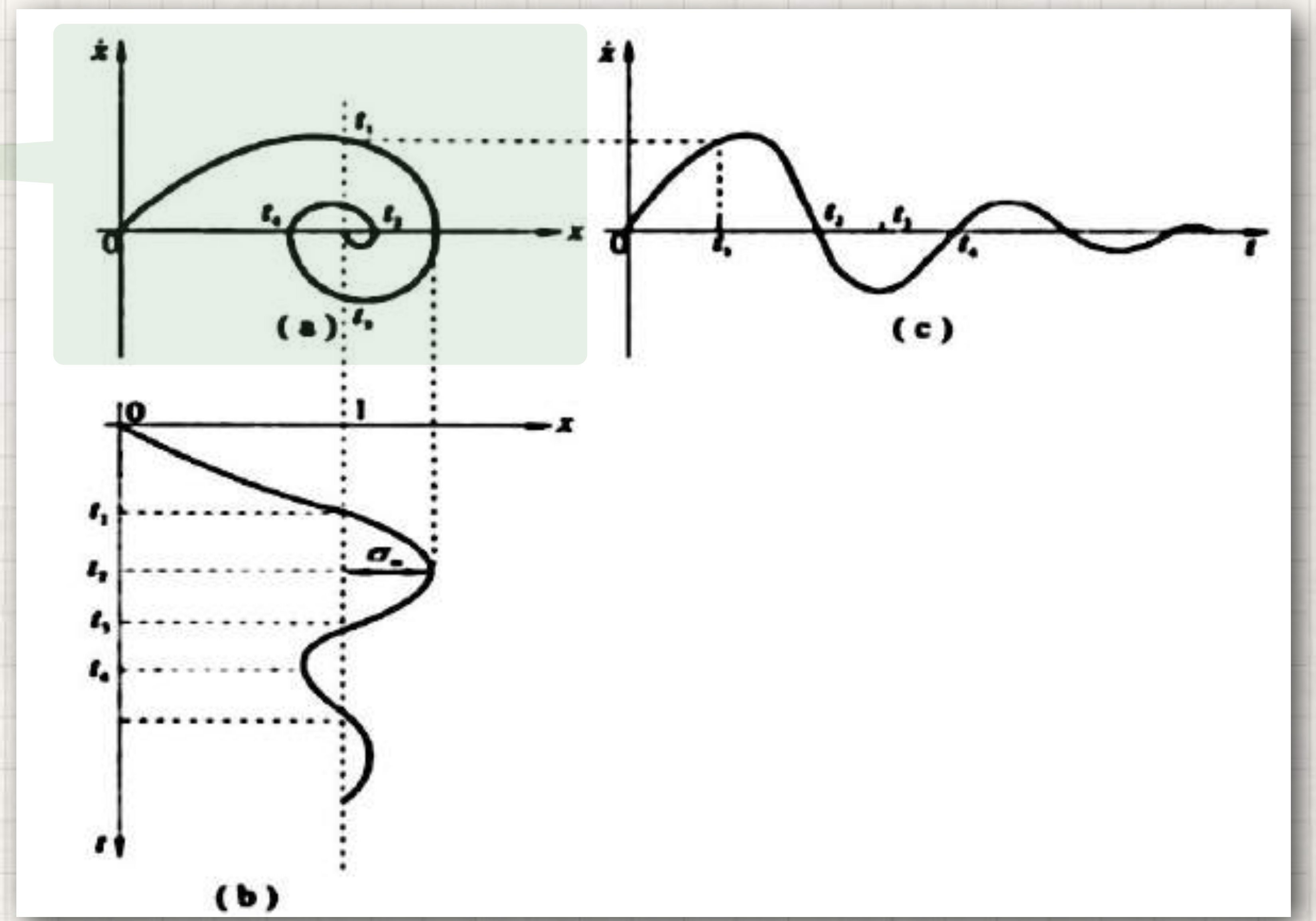
- **The curve of the solution x(t) for t≥0 passing by xo is the trajectory or orbit of equations (1) & (2).**

- **A family of those trajectories (different xo) is called the *phase portrait*.**

Concepts of Phase Plane Analysis [5]

See for example

Phase plane



It is a phase plane having $x, \dot{x}$ as their coordinates.

As time varies $t = 0 \rightarrow \infty$ change in the state of the system in $x - \dot{x}$ plane by the motion of the point.

Concepts of Phase Plane Analysis [1]

- A trajectory gives only the *Qualitative* but not the *Quantitative* behavior of the associated solution.
- For Example a closed trajectory shows that there is a periodic solution, and thus the system has sustained oscillations. Whereas, a shrinking spiral shows a decaying oscillation.

Singular Points [1]

- A singular point is an equilibrium point in the phase plane.
- Since an equilibrium point is defined as a point where the system states can stay forever, this implies that $\dot{x} = 0$ and using equations (1) and (2):

$$f_1(x_1, x_2) = 0,\ f_2(x_1, x_2) = 0 \qquad (3)$$

- The values of the equilibrium can be solved from (3).
- For a linear system, there is usually only one singular point (in some cases there can be a continuous set of singular points, as in the system $\ddot{x} + \dot{x} = 0$ for which all the points on the real axis are singular points).
- However, a nonlinear system often has more than one isolated singular point.

Phase line of a First Order System

Example 1 [*]: $\dot{x} = f(x)$, $f(x) = ax$ a is a constant, $a \neq 0$.

Determine the stability of the system analytically and then graphically.

Analytically:

The solution of the ODE is $x(t) = x_o e^{at}$, x_o is the initial condition.

To find the equilibrium point $f(x_e) = 0 \implies x_e = 0$

- Case 1: if a<0 and $x_o \neq 0$, then the solution $x(t) = x_o e^{at}$ is exponential decay and $\lim_{t\to\infty} x(t) = x_e$ the equilibrium is stable.

- Case 2: if a>0 and $x_o \neq 0$, then the solution $x(t) = x_o e^{at}$ is exponential growth and $\lim_{t\to\infty} x(t) = \infty$ the equilibrium is unstable.

[*]: Nykamp DQ, "The stability of equilibria of a differential equation." From Math Insight. http://mathinsight.org/stability_equilibria_differential_equation

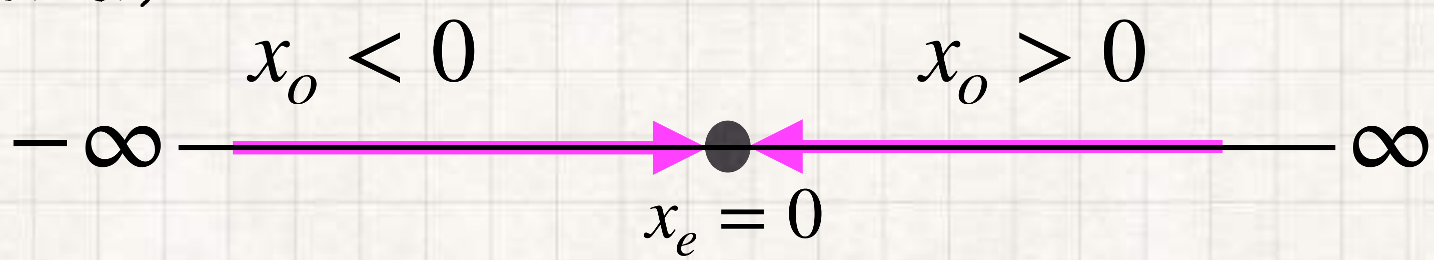
Phase line of a First Order System

Example 1 [*]: $\dot{x} = ax, a \neq 0.$

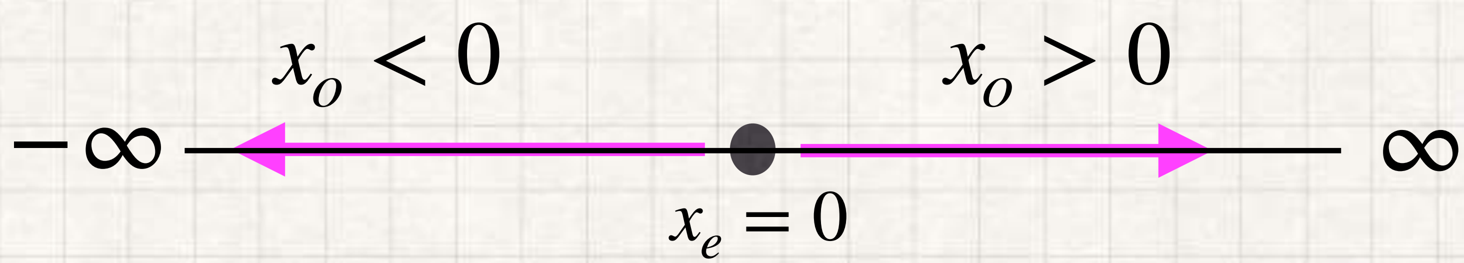
<u>Graphically:</u>

the system has only one state (first order):

Case 1: if a<0



Case 2: if a>0



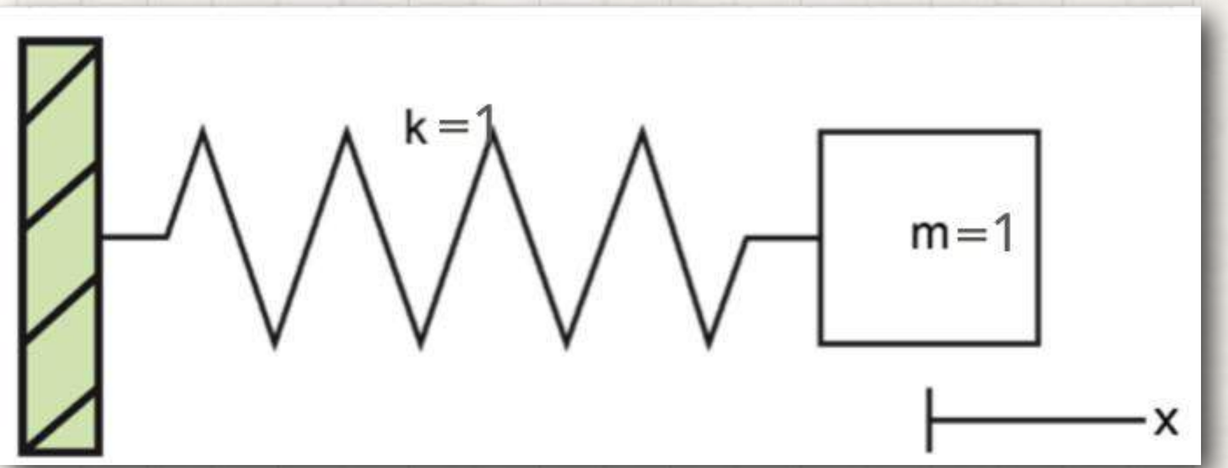
Phase diagram (LINE)

Arrows pointing towards an equilibrium solution from both sides on a phase line indicate that equilibrium solution is asymptotically stable. Arrows pointing away from an equilibrium solution from both sides on a phase line indicate that equilibrium solution is unstable.

Phase Plane of a Second Order System [1]

Example 1: Draw the Phase portrait of the following mass-spring system.

$$\ddot{x} + x = 0$$



Sol. let x1=x

$$\dot{x}_1 = x_2 \Rightarrow \dot{x}_2 = -x_1$$

The solution of the above equation is

$$x_1(t) = c_1 \cos t + c_2 \sin t \quad , \quad x_2(t) = -c_1 \sin t + c_2 \cos t$$

PROVE!

c1 and c2 are constants. Assume that the mass is initially at rest $\dot{x}(0) = 0$, at $x(0)$ length

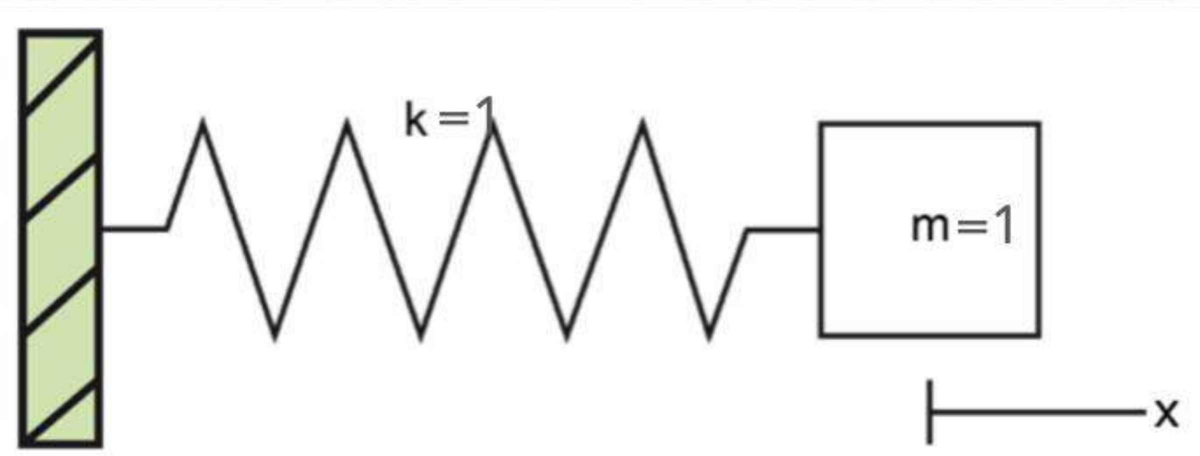
$$x_1(0) = c_1 \cos(0) + c_2 \sin(0) \Rightarrow x_1(0) = c_1 \Rightarrow x_1(t) = x_1(0) \cos t$$

$$x_2(0) = -c_1 \sin(0) + c_2 \cos(0) = 0 \Rightarrow c_2 = 0 \Rightarrow x_2(t) = -x_1(0) \sin t$$

Phase Plane of a Second Order System [1]

Example 1: Continued

Sol.



Then the solution of the above equation is

$$x_1(t) = x_1(0)\cos t$$
$$x_2(t) = -x_1(0)\sin t$$

eliminating the time from the above equations yields the equation of trajectories

$$x_1^2 + x_2^2 = x_1(0)^2 \qquad (*)$$

PROVE!

which is the equation of a circle centered at the origin and of radius xo.

Before plotting the phase portrait, locate the equilibrium point which is (0,0).

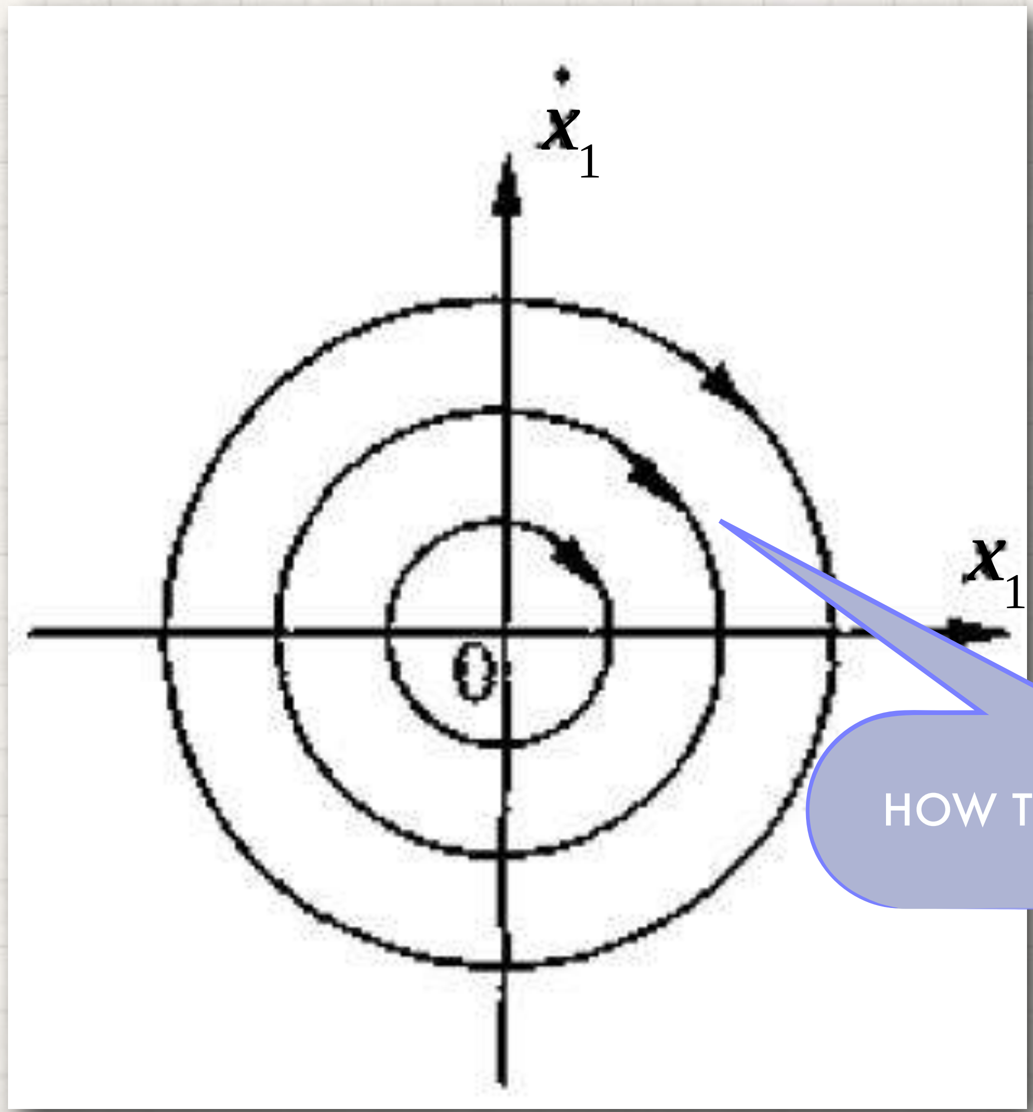
Now drawing equation (*) from different values of $x_1(0)$ will result in the phase portrait of the mass-spring system.

Phase Plane of a Second Order System [1]

Example 1: End

locate the equilibrium point which is (0,0).

Now drawing equation (*) from different values of $x_1(0), x_2(0)$ will result in the following phase portrait of the mass-spring system.



BUT
HOW TO FIND THE MOTION DIRECTION???

*The system is undamped, its eigenvalues:

$$\lambda_{1,2} = \pm i$$

HOMEWORK: DRAW THE PHASE PORTRAIT OF THIS EX. USING XO=(1,1),(-2,1), (-3,-2).
WHAT HAPPENS IF XO=(0,0)?

Nonlinear Systems Analysis

LECTURE 5

4th GRADE-CONTROL AND SYSTEMS ENGINEERING

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T. MOHAMMADRIDHA

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Drawing Phase Portrait [1]

- Today, phase portraits are routinely computer-generated. In fact, the associated ease of quickly generating phase portraits allowed many advances in the study of complex nonlinear dynamic behaviors such as chaos.

- It is still practically useful to learn how to roughly sketch phase portraits or quickly verify the plausibility of computer outputs (just like in the case of root locus for linear systems).

- We will use phase portrait to study the behavior of the system near equilibrium (singular) points.

Drawing methods [1]

They are number of methods to draw the phase trajectories for linear and nonlinear systems. Here we will discuss two: the Analytical method and the Graphical method.

- The analytical method involves the analytical solution of the differential equations describing the systems.

  - It is based on eliminating the time variable either after solving for x1(t) an x2(t) or by solving $\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}$. Both techniques lead to a functional relation between the two phase variables X1 and x2 to generate the phase portrait.

  - This method can be applied when the differential equation is relatively simple to solve. It is useful for linear and some special nonlinear systems, particularly piece-wise linear systems, whose phase portraits can be constructed by piecing together the phase portraits of the related linear systems (see ex. 2.5 in [1]).

Drawing methods:Graphical [1,4]

The graphical method is used to construct the phase trajectories indirectly.

There are many graphical methods to sketch the phase portrait.

- One widely used method is the method of isoclines (will be shown later).

- Another simple method is to construct trajectories from the vector field diagram [4].
    - The vector field at a point is tangent to the trajectory through that point.

Phase Portrait: Vector field [4]

Rewrite the autonomous system in (1),(2) as:

$$\dot{x} = f(x) \qquad (3)$$

- where f(x) is the vector ($f_1(x)$,$f_2(x)$).
- Consider f(x) as the vector field in the state plane: at each point in the plane x we assign vector f(x). (x a point, $\dot{x}$ velocity vector at that point)
- We can also visualize the vector as a directed line $\longrightarrow$ segment from any initial point P_1 to a final point P_2. Then, the vector from x=P_1 to x=P_2 is given by: $\overrightarrow{V} = P_2 - P_1$
- Here our vector is $\overrightarrow{V} = f(x) = P_2 - P_1$

Concepts of Phase Plane Analysis [4]

- Here our vector is

$$\boxed{\dot{x} = f(x) = P_2 - P_1}$$

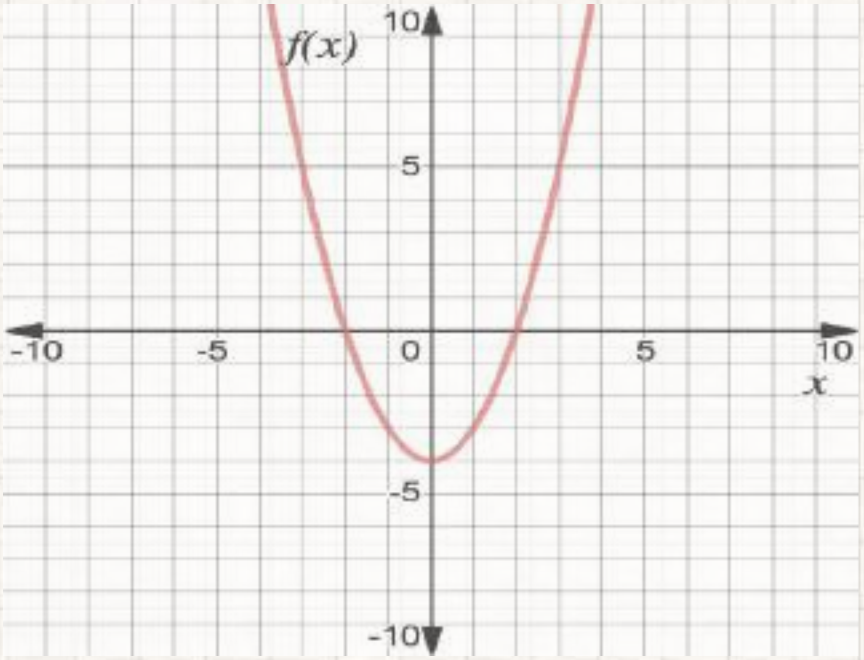
- We represent $f(x)$ as a vector based at x=P1, i. e. we assign to x the directed line segment

- $$P_2 = x + f(x), x = P_1 \implies P_2 = P_1 + f(P_1)\,.$$

Phase line of First Order Systems

Example 2: For $\dot{x} = f(x)$, $f(x) = -4 + x^2$ use phase diagram to analyze its stability.

Consider the rate of change f(x) we notice that it is increasing whenever $x^2 > 4$



Sol. The equilibrium point(s) are found as follows: $\dot{x} = f(x_e) = 0$

- $$x_e^2 - 4 = 0 \implies x_{e_1} = -2, \; x_{e_2} = 2$$

Phase line of First Order Systems

Example 2: For $\dot{x} = f(x)$, $f(x) = -4 + x^2$ use phase diagram to analyze its stability.

Sol. stability be examined by drawing the vector diagram from a point x=P1:

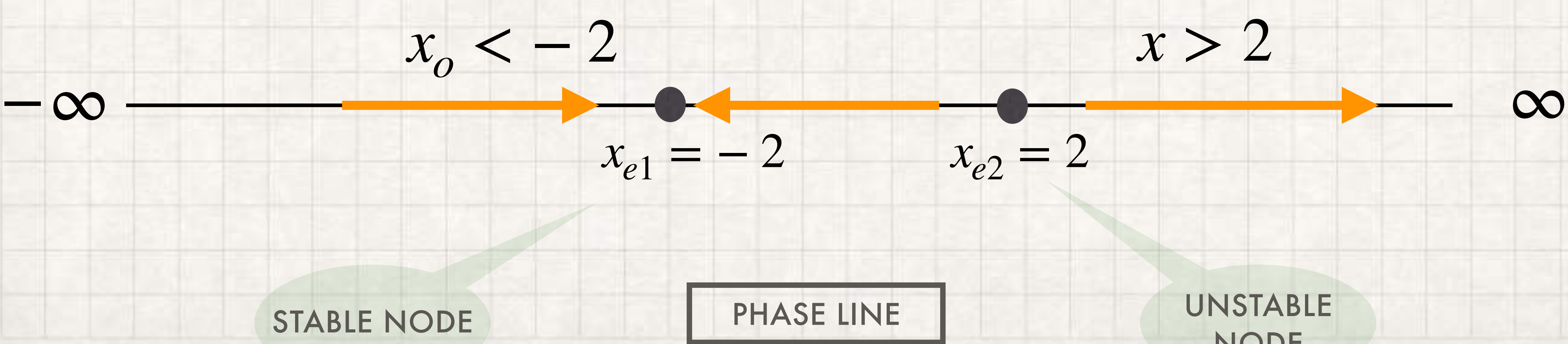
$$P_2 = P_1 + f(P_1)$$

choose few points around equilibrium points:

$$P_1 = 1 \implies P_2 = 1 + f(1) = 1 - 3 = -2$$

$$P_3 = 4 \implies P_4 = 4 + 12 = 16$$

$$P_5 = -3 \implies P_6 = -3 + f(-3) = -3 + 5 = 2$$



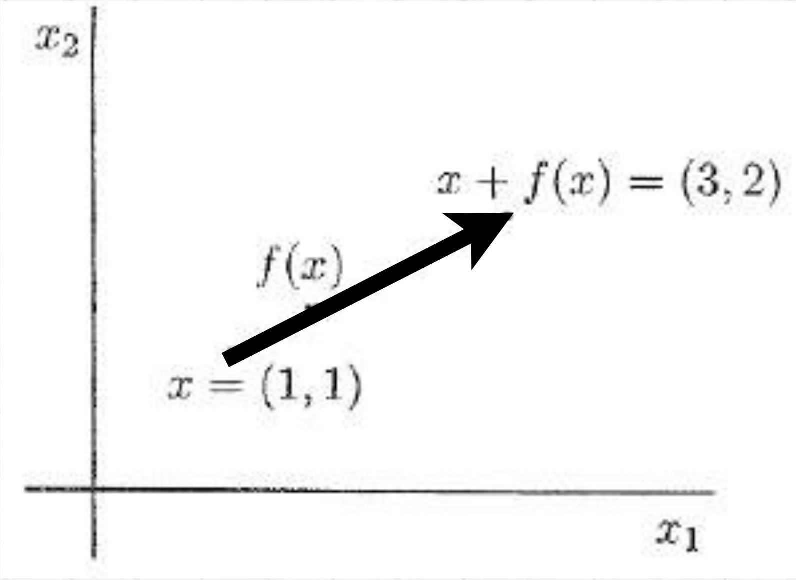
Vector field for second order system [4]

- Example $\dot{x} = f(x) = (f_1(x), f_2(x)) = (2x_1^2, x_2)$

To draw the vector at x=(1,1):

we draw an arrow pointing from P_1=x=(1,1) to

P_2= P_1+($f_1(P_1)$,$f_2(P_1)$) =(1,1)+(2,1)=(3,2).



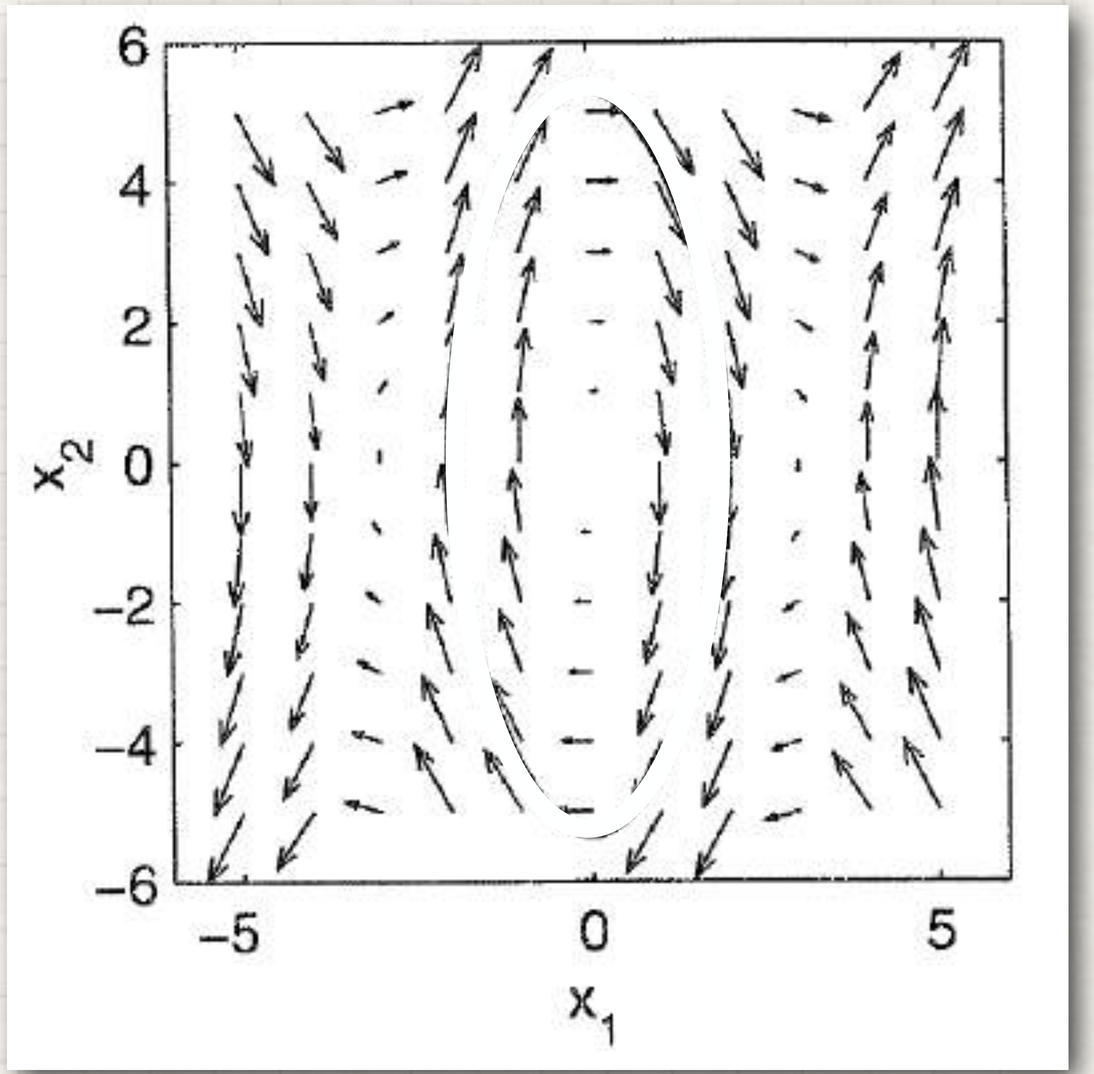
- Repeating this for every point we obtain a vector field diagram (see page 36-37 of [4]).

Graphical phase portrait for second order systems [4 P36-37]

- The length of the arrow at a given point is proportional to the length of $f(x)$ i.e. $\sqrt{f_1^2(x)+f_2^2(x)}$.

- For convenience, we draw arrows of equal length at all points.

- The *vector field* at a point is ***tangent*** to the trajectory through that point.

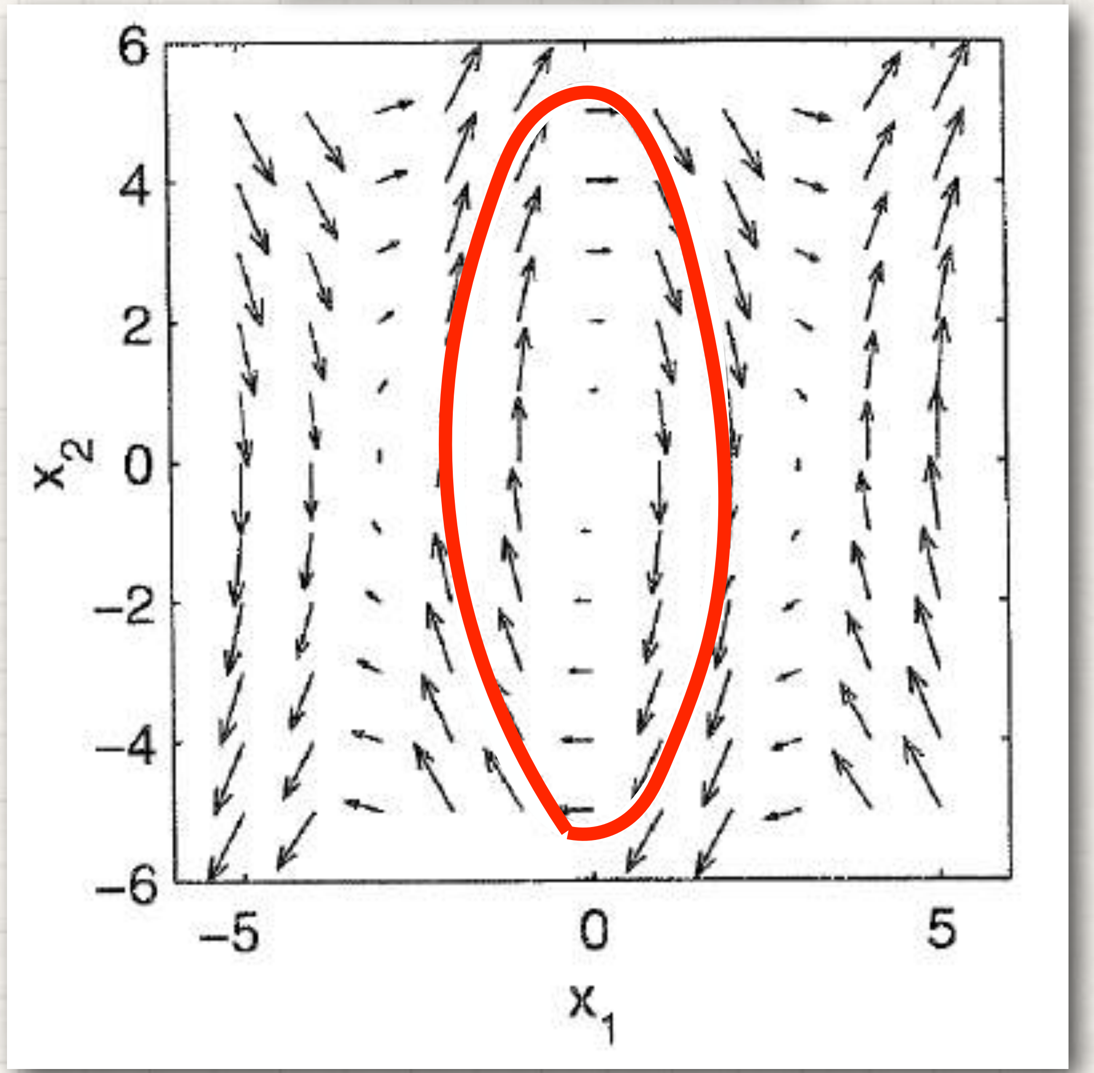


Vector field diagram of the pendulum equation without friction.

Graphical phase portrait for second order systems [4 P36-37]

- We can, in essence, construct trajectories from the vector field diagram.
- Starting a trajectory from $x_o = (x_1(0), x_2(0))$, it moves along the vector field at x_o. This will lead us to a new point x_a.
- We continue the trajectory along the vector field at x_a.
- If this process is repeated carefully and consecutive points are chosen close to each other, we can obtain a reasonable approximation of the trajectory through *xo*.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin x_1 \end{aligned}$$



Vector field diagram of the pendulum equation without friction.

✸ a closed trajectory indicates periodic solution.

Drawing Phase Portrait [6]

- Phase plane trajectories follow the direction field. The velocity vector for a solution at a point (x1, x2) in the plane is f1(x1,x2), f2(x1,x2) . The direction of the trajectory is the direction of this vector.
- An approximate picture of the phase portrait can be constructed by plotting trajectories from a large number of initial states spread all over the state plane.
- The curves f1(x1,x2)=0 and f2(x1,x2)=0 are the nullclines on which the direction of a trajectory is vertical and horizontal respectively.
- The intersection points of the nullclines represent the equilibrium points.

Phase Portrait [4, 6]

- Phase plane portrait can be easily constructed using computer simulations.

- Since the time t is eliminated in a trajectory, it is not possible to rebuild the solution ($x1(t)$, $x2(t)$) associated with a given trajectory. Therefore, a trajectory provides a qualitative but not quantitative behavior for the associated solution.

- For example, a closed trajectory indicates periodic solution, i.e. sustained oscillation while a shrinking spiral indicates a decaying solution.

PHASE PLANE ANALYSIS FOR

LINEAR AUTONOMOUS SYSTEMS

Phase Plane of Linear Systems [4]

- The phase portrait provides important information about system stability throughout the behavior around each equilibrium point.
- We'll describe the phase plane analysis of linear systems because nonlinear systems behave similarly to a linear system around equilibrium points.
- A nonlinear system near an equilibrium point can take one of the patterns of linear systems.
- Correspondingly the equilibrium points are classified as stable node, unstable node, saddle, stable focus, unstable focus, or center.
- In general, qualitative behavior of a nonlinear system near an equilibrium point can be determined via <u>linearization</u> around that equilibrium point.

Phase Plane of Linear Systems [1]

- We will classify the type and stability of the equilibrium solution of a given linear system by the shape formed by the trajectories about each critical point.

  We will simply consider the second-order linear system described by

$$\dot{x} = Ax$$

- $\boxed{v_1, v_2}$ are the eigenvectors associated to the eigenvalues $\boxed{\lambda_1, \lambda_2}$ respectively.
- To obtain the phase portrait of this linear system, we first solve for the time history x(t), e.g.:

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \quad for \quad \lambda_1 \neq \lambda_2$$

c_1, c_2 are constants (scalars), $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Phase Plane of Linear Systems [1]

- The following cases of system (nonzero) eigenvalues can occur

1. λ_1 and λ_2 are both real and have the same sign (positive or negative)
2. λ_1 and λ_2 are both real and have opposite signs
3. λ_1 and λ_2 are complex conjugate with non-zero real parts
4. λ_1 and λ_2 are complex conjugates with real parts equal to zero

Phase Plane of Linear Systems [9]

Given $x' = Ax$, where there is only one critical point, at (0,0):

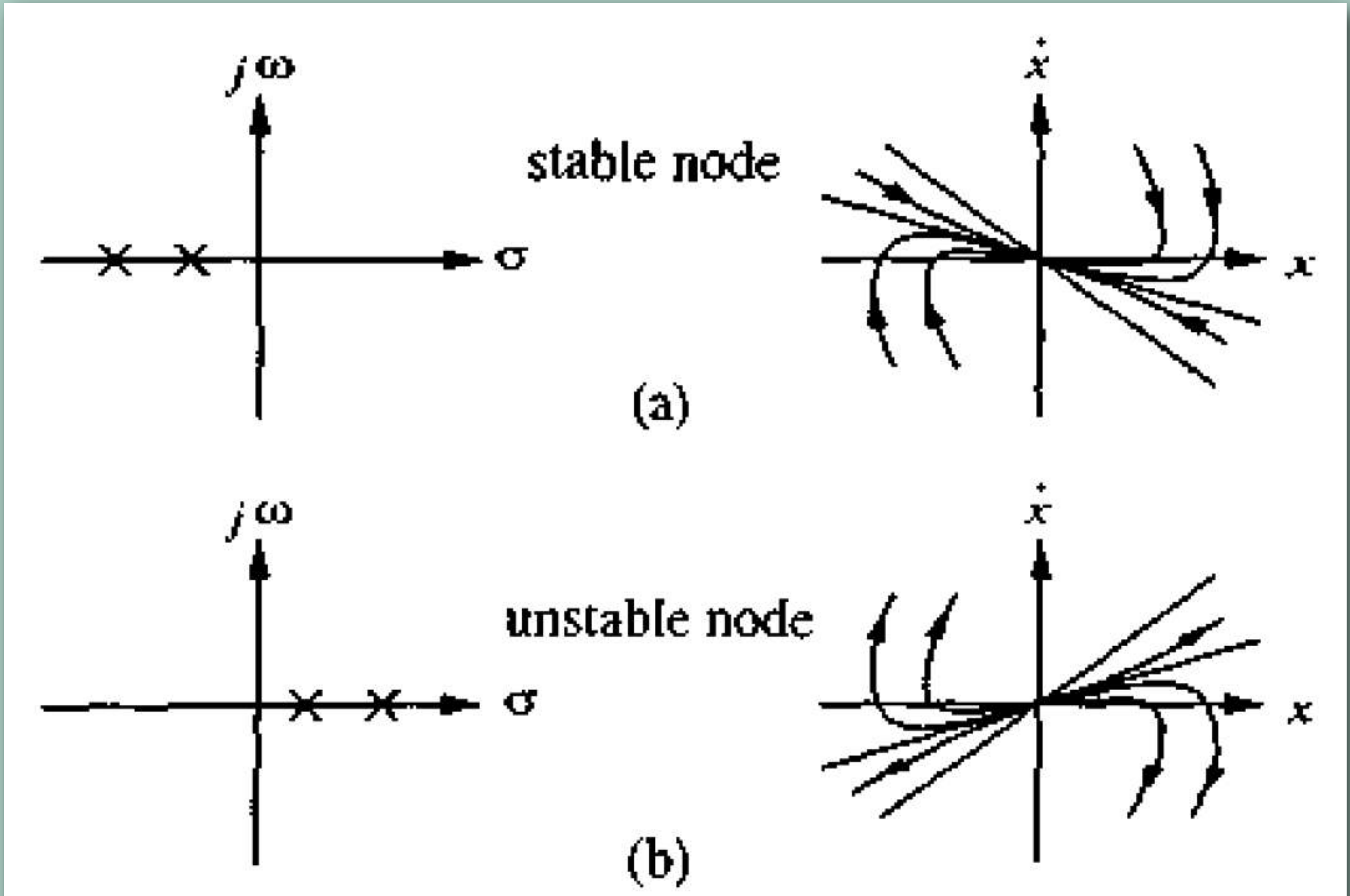
- CASE 1: Real Distinct eigenvalues

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 \quad for \quad \lambda_1 \neq \lambda_2$$

- When λ_1, λ_2 are both positive, or are both negative
- The trajectories that are the eigenvectors move in straight lines.
- The rest of the trajectories would bend toward the direction of the eigenvector of the eigenvalue with the larger value.
- The trajectories either move away from the singular point to infinite-distant away (when λ_1, λ_2 are both positive).
- Or move toward from infinite-distant and converge to the equilibrium point (when λ_1, λ_2 are both negative).
- This type of critical point is called a node. It is asymptotically stable if the eigenvalues are both negative, unstable if both are positive.

Phase Portrait of Linear Systems [1]

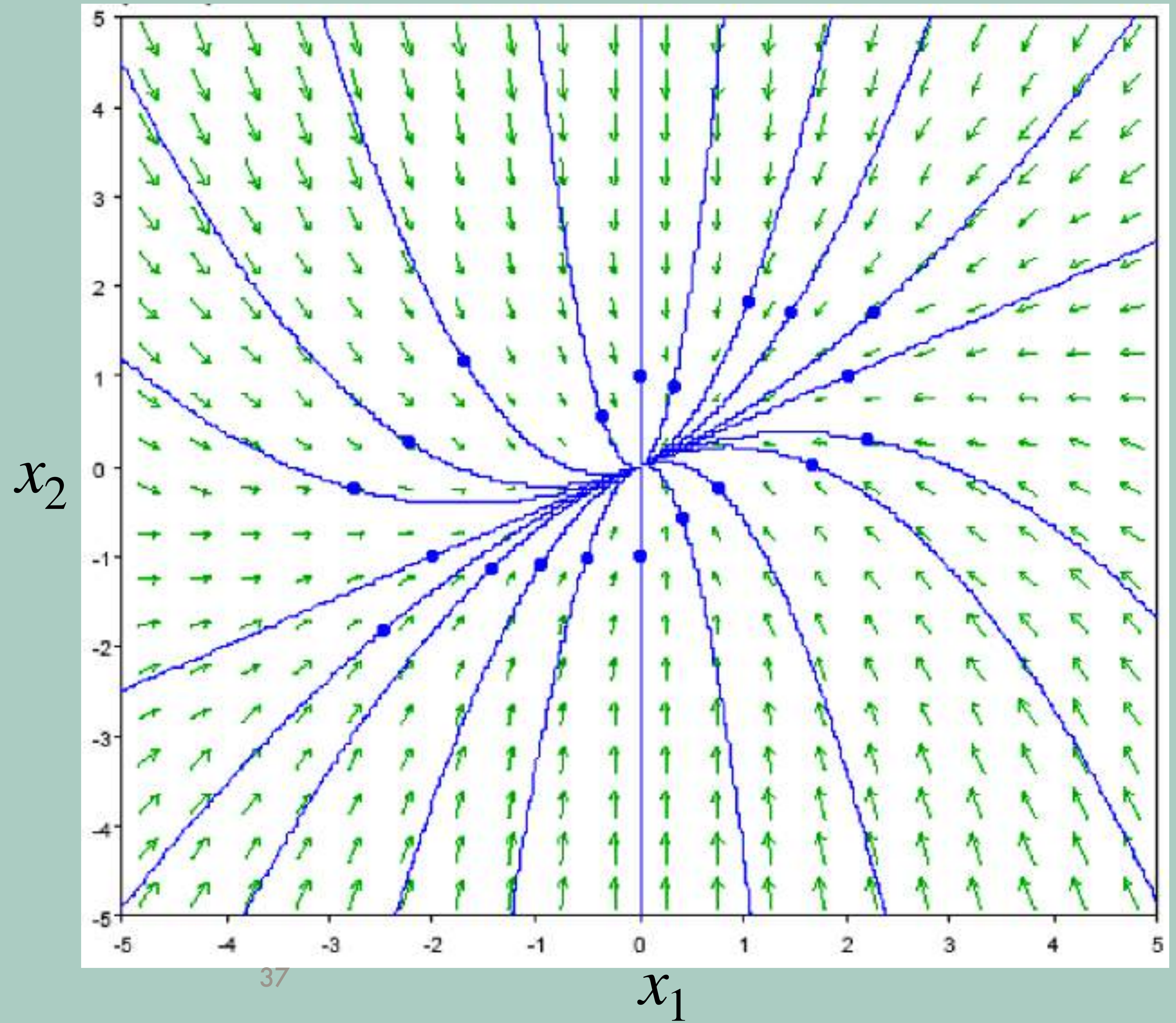
- The trajectories that are the eigenvectors move in straight lines toward the equilibrium or away from it depending on the signs of the eigenvalues.



Phase Portrait of Linear Systems [1]

- **Example:** $\dot{x}_1 = -2x_1 \quad \dot{x}_2 = x_1 - 4x_2$

- **Equilibrium stability:** Stable node



Phase Plane of Linear Systems [9]

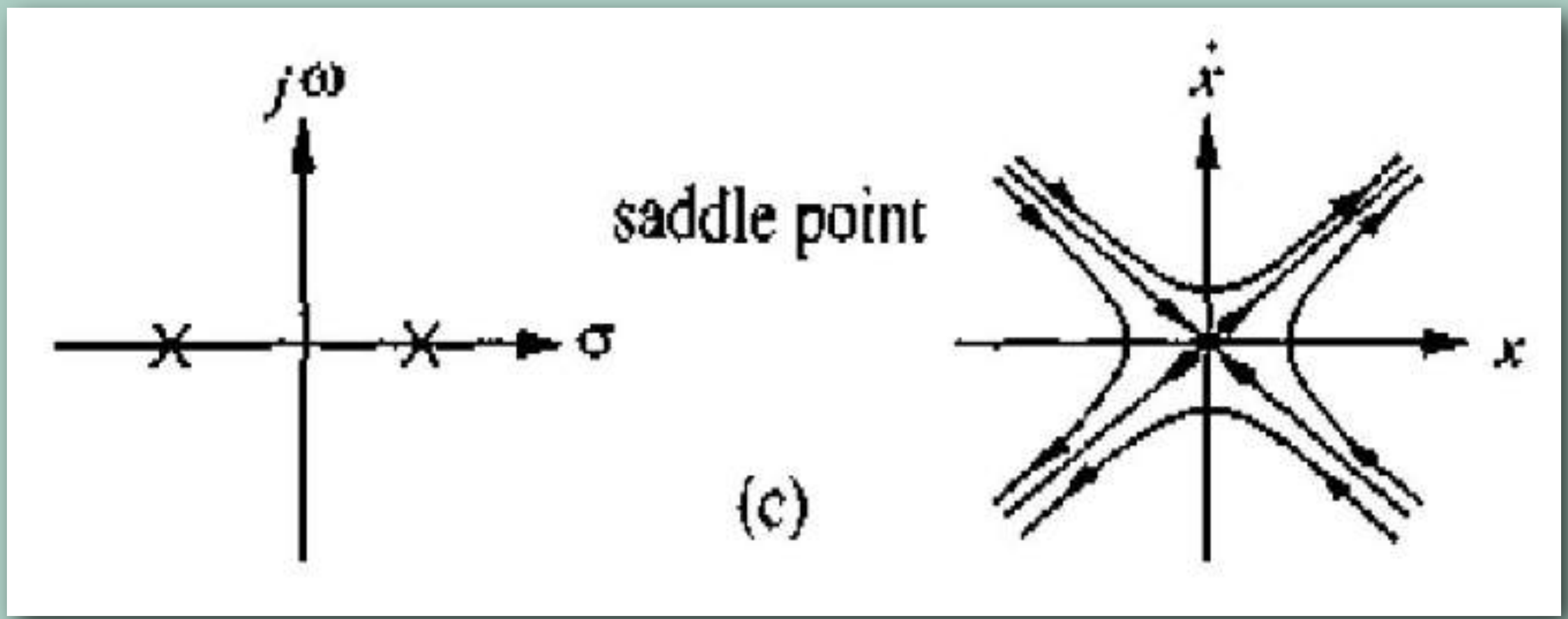
Given $x' = Ax$, where there is only one critical point, at (0,0):

- CASE 1: Real Distinct eigenvalues

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 \quad for \quad \lambda_1 \neq \lambda_2$$

- When λ_1, λ_2 have opposite signs
- the trajectories given by the eigenvectors of the negative eigenvalue initially start at infinite-distant away, move toward and eventually converge at the critical point.
- The trajectories that represent the eigenvectors of the positive eigenvalue move in exactly the opposite way: start near the critical point then diverge to infinite-distant out.
- Every other trajectory starts at infinite-distant away, moves toward but never converges to the critical point, before changing direction and moves back to infinite-distant away.
- This type of critical point is called a saddle point. It is always unstable.

Phase Portrait of Linear Systems [1]



Phase Plane of Linear Systems [9]

Given $x' = Ax$, where there is only one critical point, at (0,0):

CASE 2: Real repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

- If the matrix A is a multiple of the Identity matrix then there are two linearly independent eigenvectors

$$A = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha \text{ is any nonzero constant,} \quad x(t) = e^{\lambda t}(C_1 v_1 + C_2 v_2)$$

Every nonzero solution traces a straight-line trajectory, in the direction given by the vector C1 v1 + C2 v2. The phase portrait thus has a distinct star-burst shape.

PROVE IT

- the equilibrium point is a **proper node or star node.**

Stability: It is unstable if the eigenvalue is positive; asymptotically stable if the eigenvalue is negative.

Phase Plane of Linear Systems [9]

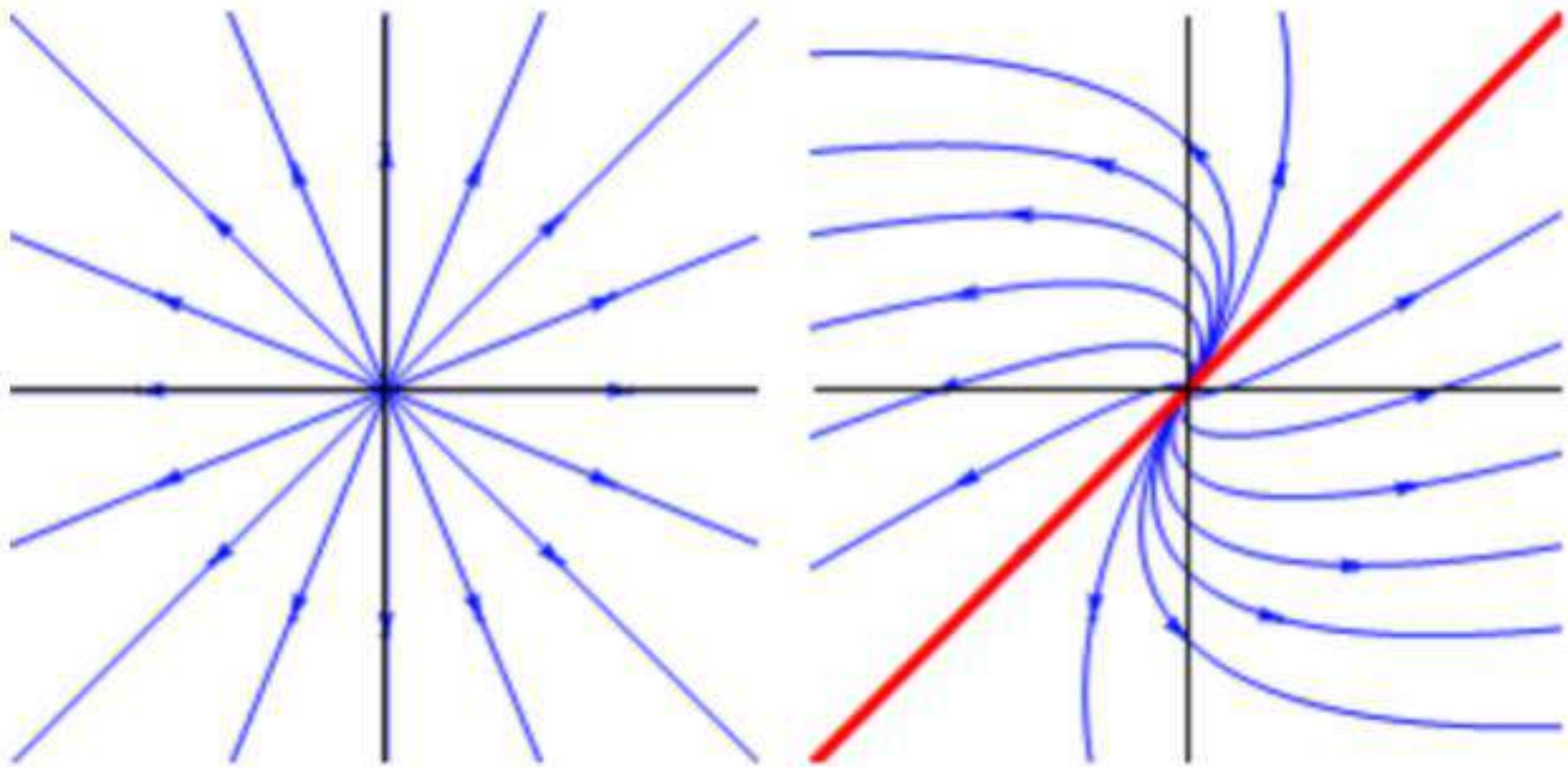
Given $x' = Ax$, where there is only one critical point, at (0,0):

- CASE 2: Real repeated eigenvalues, $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, $A \neq \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - If the eigenvalues are real and repeated, then the critical point is either **a star** or it is an **improper node.**
  - If the matrix A is not a multiple of the Identity matrix and there is one linearly independent eigenvector v_1:

$$x(t) = C_1 e^{\lambda t} v_1 + C_2 e^{\lambda t}(t v_1 + v_2)$$

  - In this case of improper node, trajectories are tangential to the sole eigenvector.
  - It is asymptotically unstable if $\lambda > 0$, stable if $\lambda < 0$.

Phase Portrait of Linear Systems [8]



A star (unstable) and an improper node (unstable)

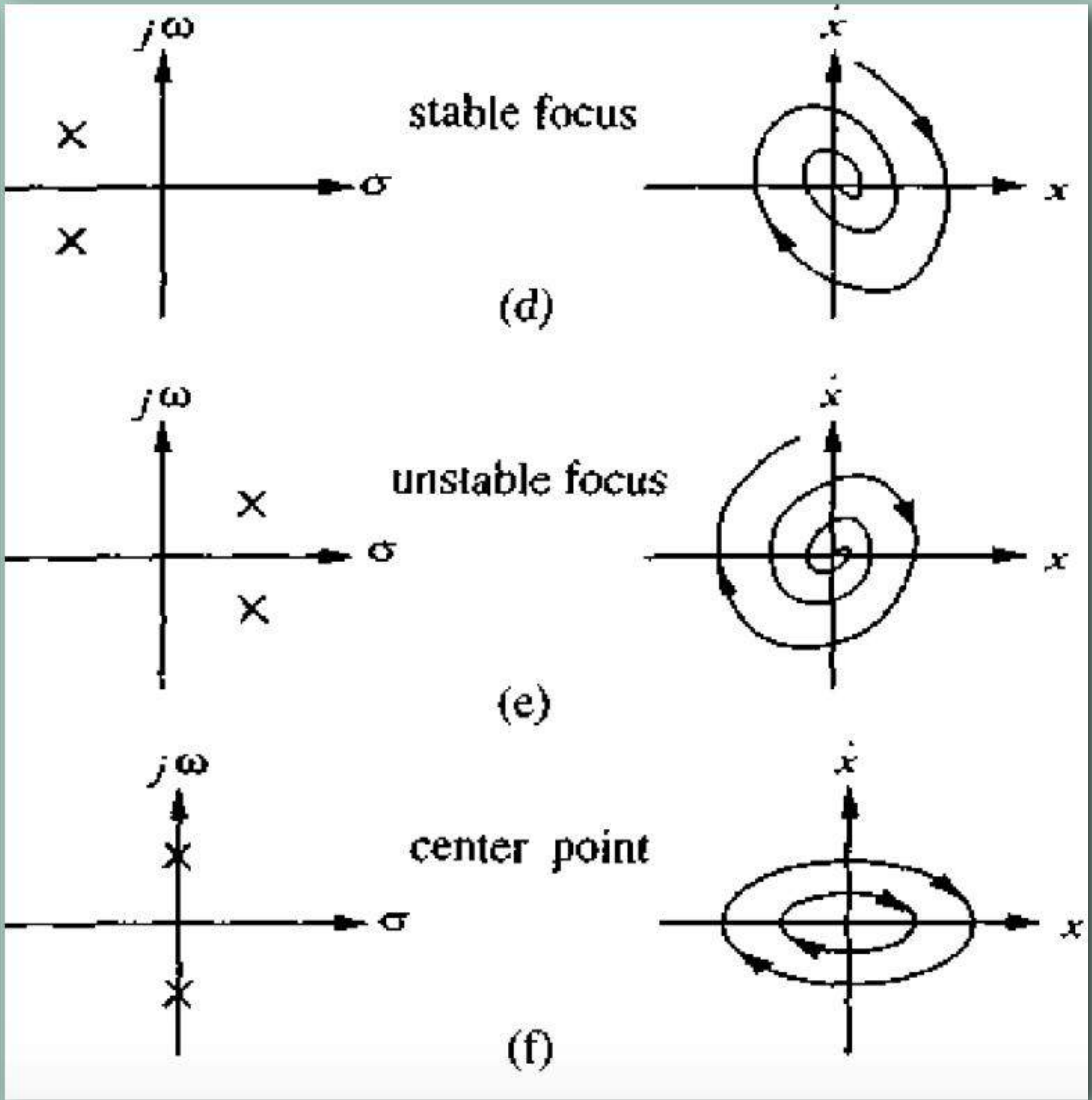
Phase Plane of Linear Systems [9]

Given $x' = Ax$, where there is only one critical point, at (0,0):

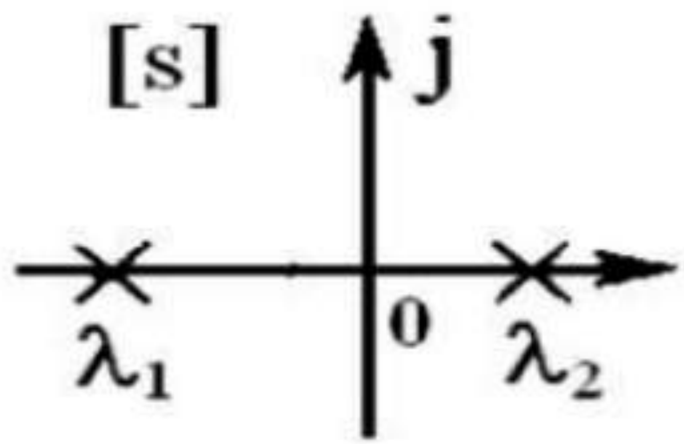
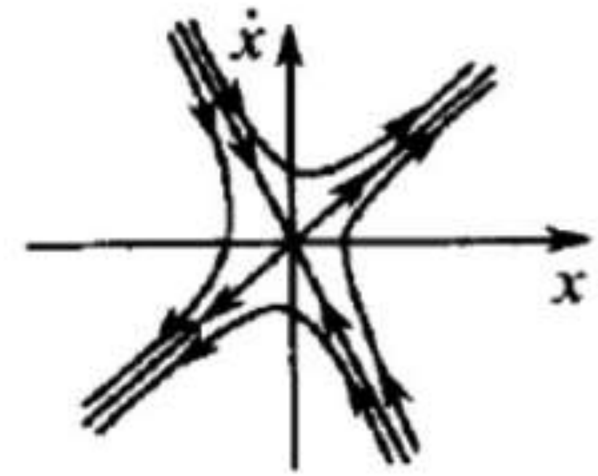
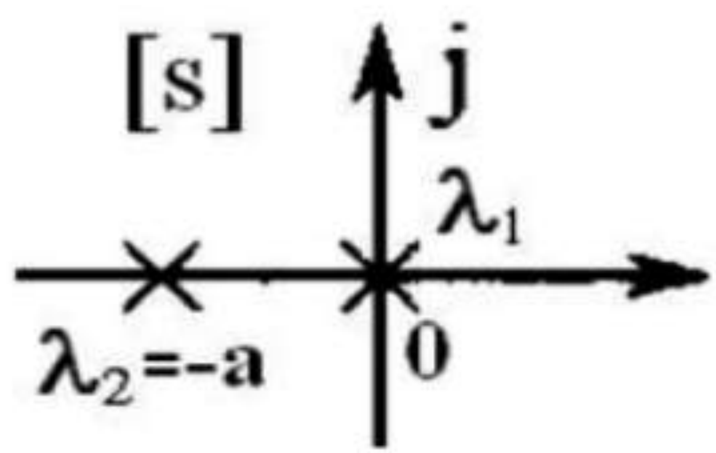
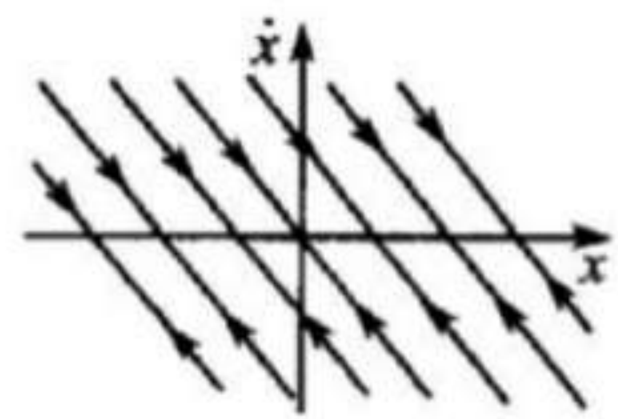
- CASE 3: Complex conjugate eigenvalues, $\lambda_{1,2} \in \mathbb{C}$

  - If the eigenvalues are non-real of the form $\lambda_{1,2} = \alpha \pm \beta i$ the critical point is either a spiral point or a center point.
  - If $\alpha > 0$, the critical point is an unstable focus (spiral) point.
  - If $\alpha < 0$, the critical point is an asymptotically stable focus (spiral) point.
  - If $\alpha = 0$, the critical point is a center and sometimes it is referred to as neutrally stable.

Phase Portrait of Linear Systems [8]



Phase Portrait of Linear Systems [7]

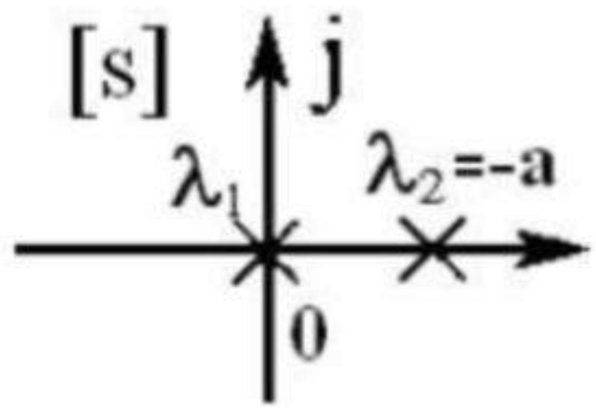
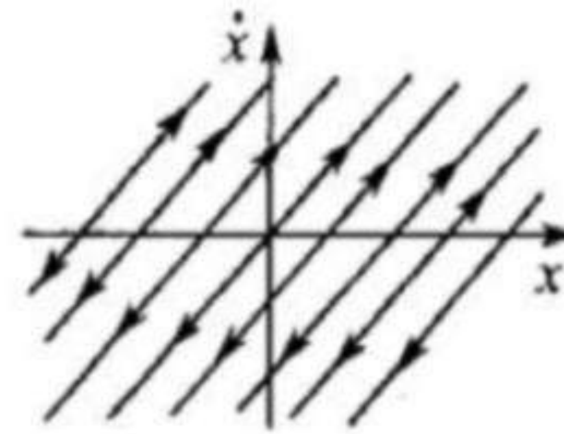
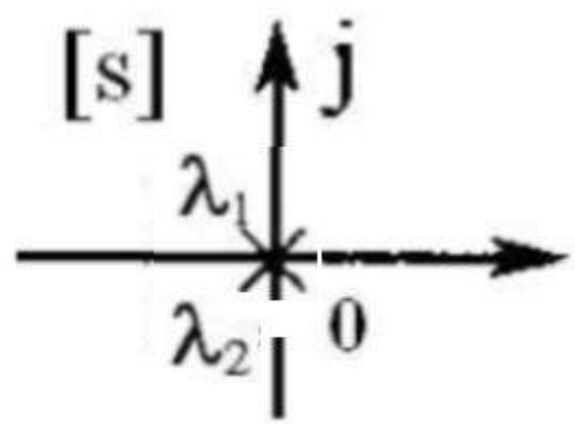
Distribution of Poles	**Phase Portrait**	**Singular Point**
		*(0,0) Saddle point*
		*X-axis*

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2, a > 0$$

Phase Portrait of Linear Systems [7]

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_2, a < 0$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

Distribution of Poles	Phase Portrait	Singular Point
		x axis
		x axis

Phase Plane of Linear Systems [9]

Stability in summary

As t increases $t \to \infty$, if all (or almost all) trajectories

- ☑ converge to the critical point → asymptotically stable,
- ☑ move away from the critical point to infinitely far away → unstable,
- ☑ stay in a fixed orbit within a finite (i.e., bounded) range of distance away from the critical point → stable (or neutrally stable).

Phase Plane Analysis of Linear systems

Example 2: Draw the Phase portrait of the following system.

$$\dot{x} = Ax, A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

Sol.

$$\dot{x}_1 = -x_1 + 4x_2$$

$$\dot{x}_2 = -2x_1 + 5x_2$$

The equilibrium point is found as follows $\dot{x} = Ax = 0 \Rightarrow x_{eq} = (0,0),$

The eigenvalues and eigenvectors are $\lambda_1 = 1,\ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $, \lambda_2 = 3, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_2 = 0.5x_1$ $\qquad$ $x_2 = x_1$

The equilibrium point is unstable node.

Draw in phase plane the eigenvectors.

REMINDER: TO FIND THE EIGENVECTOR FOR THE ABOVE SYSTEM USE $(\lambda_i I - A)\vec{v_i} = \vec{\mathbf{0}}$

Concepts of Phase Plane Analysis

Example 2:

$$\dot{x} = Ax, A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \lambda_2 = 3, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Draw the eigenvectors v1 and v2.

Choose few points like x1=2, X2=1 or P1=(2,1) and draw the resulting vector:

$$P_2 = P_1 + f(x) = P_1 + Ax$$

$$P_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Thus the vector based at (2,1) is pointing to (4,2) .. is it logical?? why??

Phase Plane Analysis of Linear systems

Example 2: $\dot{x} = Ax, A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$

x=P3=(1,1) and draw the resulting vector:

$$P_4 = P_3 + Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Thus the vector based at (1,1) is pointing to (4,4) .. why??

Choose a point not on the eigen vectors like P5=(4,1)

$$P_6 = P_5 + Ax = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

Concepts of Phase Plane Analysis

Example 2:

$$\dot{x} = Ax, A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

Now see what happens from the result P6=(4,-2)

$$P_7 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -20 \end{pmatrix}$$

- Notice that the vector field is directed toward the eigen vector v2 of $\lambda_2 = 3$
- This is because as $x(t) = C_1 e^t v_1 + C_2 e^{3t} v_2$
- $e^t < e^{3t}$ for $t \to \infty$

Phase Plane Analysis of Linear systems

Example 2:

$$\dot{x} = Ax, A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}$$

Equilibrium:

Unstable Node

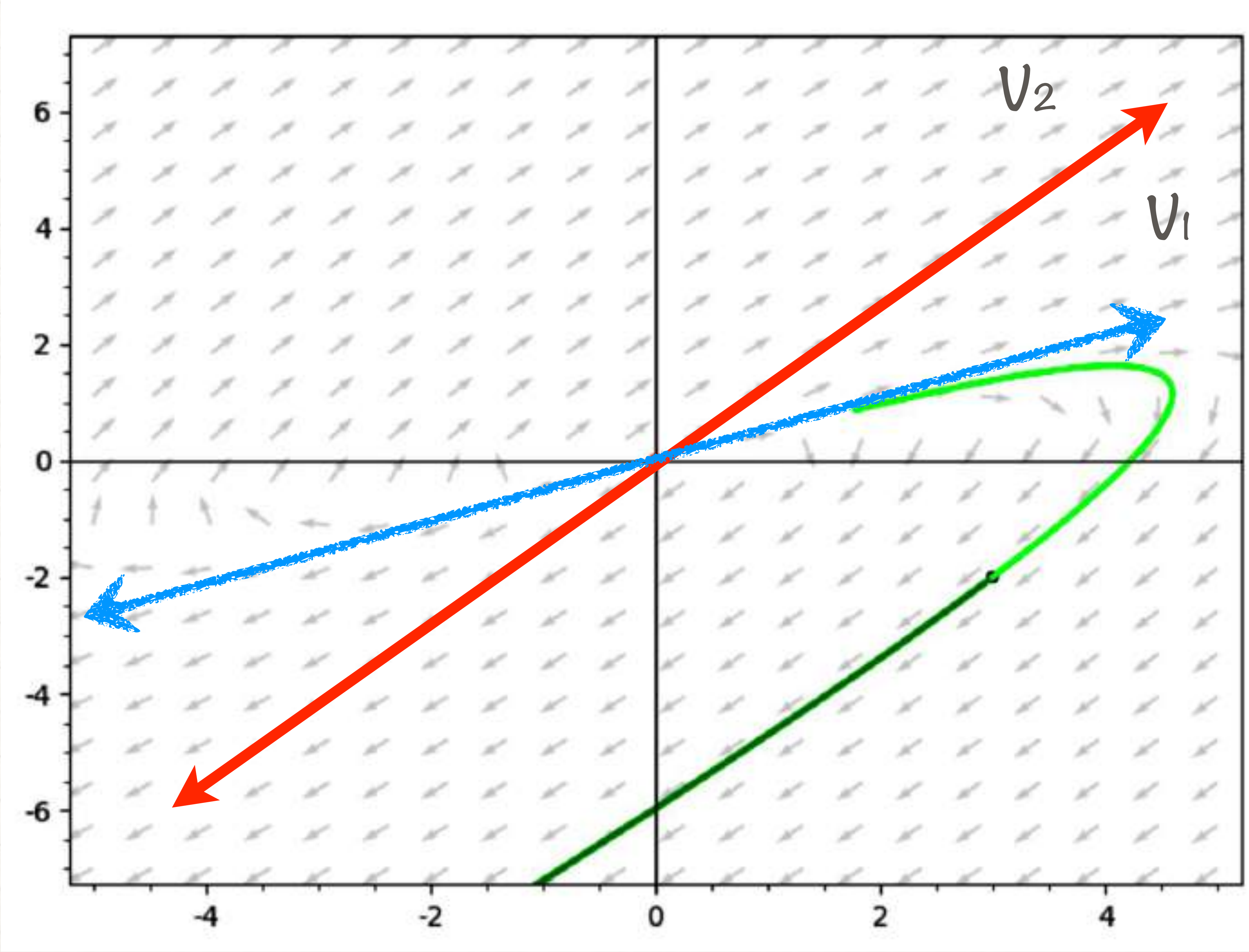
Note the the vector

field starts parallel

to V1 and ends

parallel

to V2 .



Concepts of Phase Plane Analysis

Example 3: Draw the Phase portrait of the following system.

$$\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$$

Sol.

1. The equilibrium point is $x_{eq} = (0,0)$

2. The eigenvalues and eigenvectors are $\lambda_1 = -2,\ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $,\lambda_2 = -3, \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$x_1 = 0$

$x_2 = -3x_1$

3. The equilibrium point is a <u>stable node.</u>

4. <u>Draw the eigenvectors and nullclines on the phase plane.</u>

5. <u>Now, take few points on the eigenvectors above and below each and check the direction.</u>

Phase Plane Analysis of Linear systems

Example 3:

$$\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$$

The nullclines are: $1) x_1 = 0,$ $2) x_2 = \dfrac{3}{2} x_1$

Choose few points like $x_1=-2$, $x_2=0$

$$P_2 = P_1 + Ax$$

$$P_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

Take on the second nullcline $x=(-1,-1.5)$:

$$P_3 = \begin{pmatrix} -1 \\ -1.5 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1.5 \end{pmatrix}$$

Concepts of Phase Plane Analysis

Example 3:

$$\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$$

Now see what happens from the result P6 = (4,-2)

$$P_7 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ -20 \end{pmatrix}$$

- Notice that the vector field is directed toward the eigenvector v1 of $\lambda_1 = -2$
- This is because v1 is associated to the slowest eigenvalue $\lambda_1 = -2$ why?

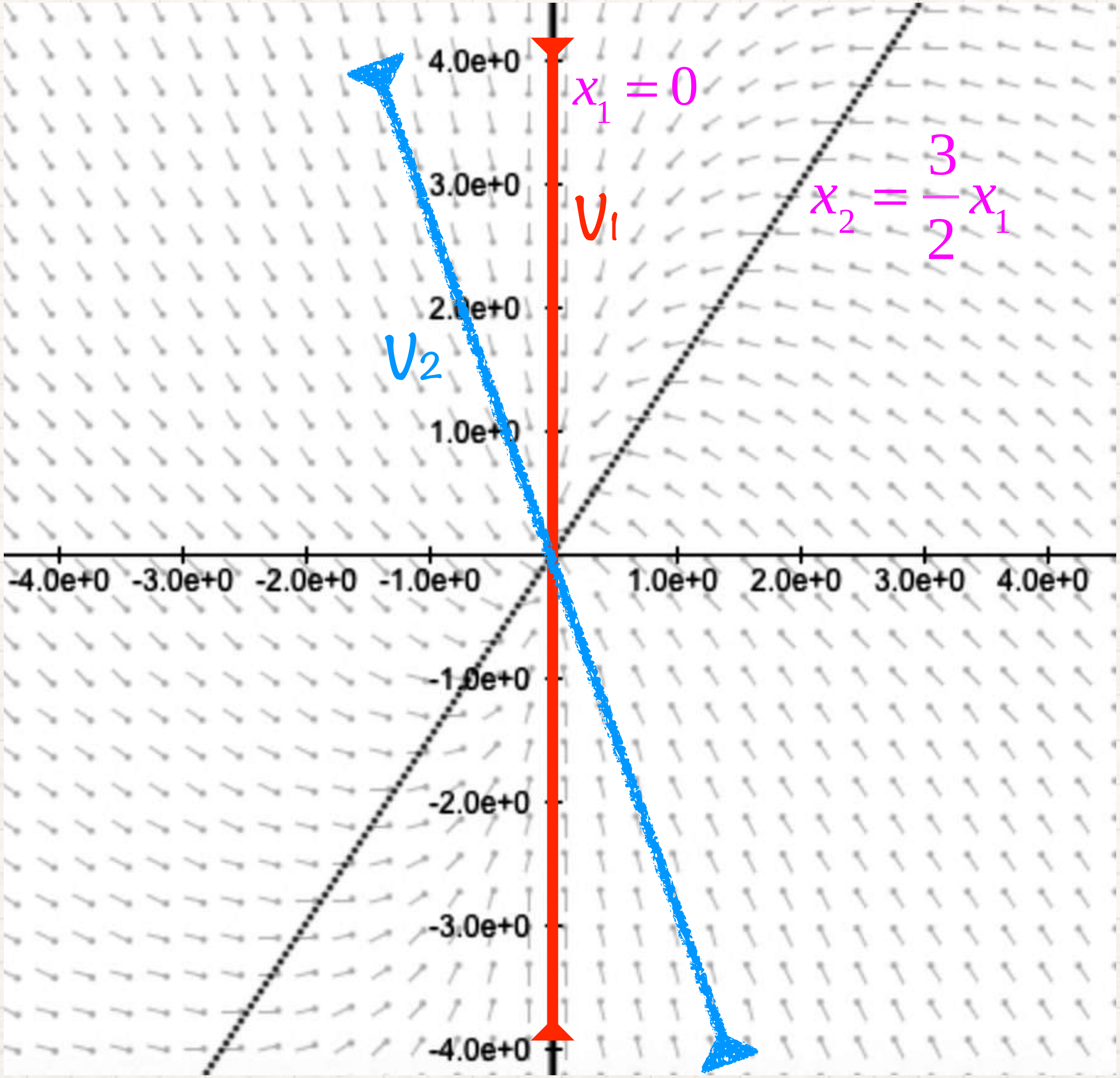
<u>See section 2.1 of [4] for details</u>.

Phase Plane Analysis of Linear systems

$$\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$$

$$\lambda_1 = -2, \ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

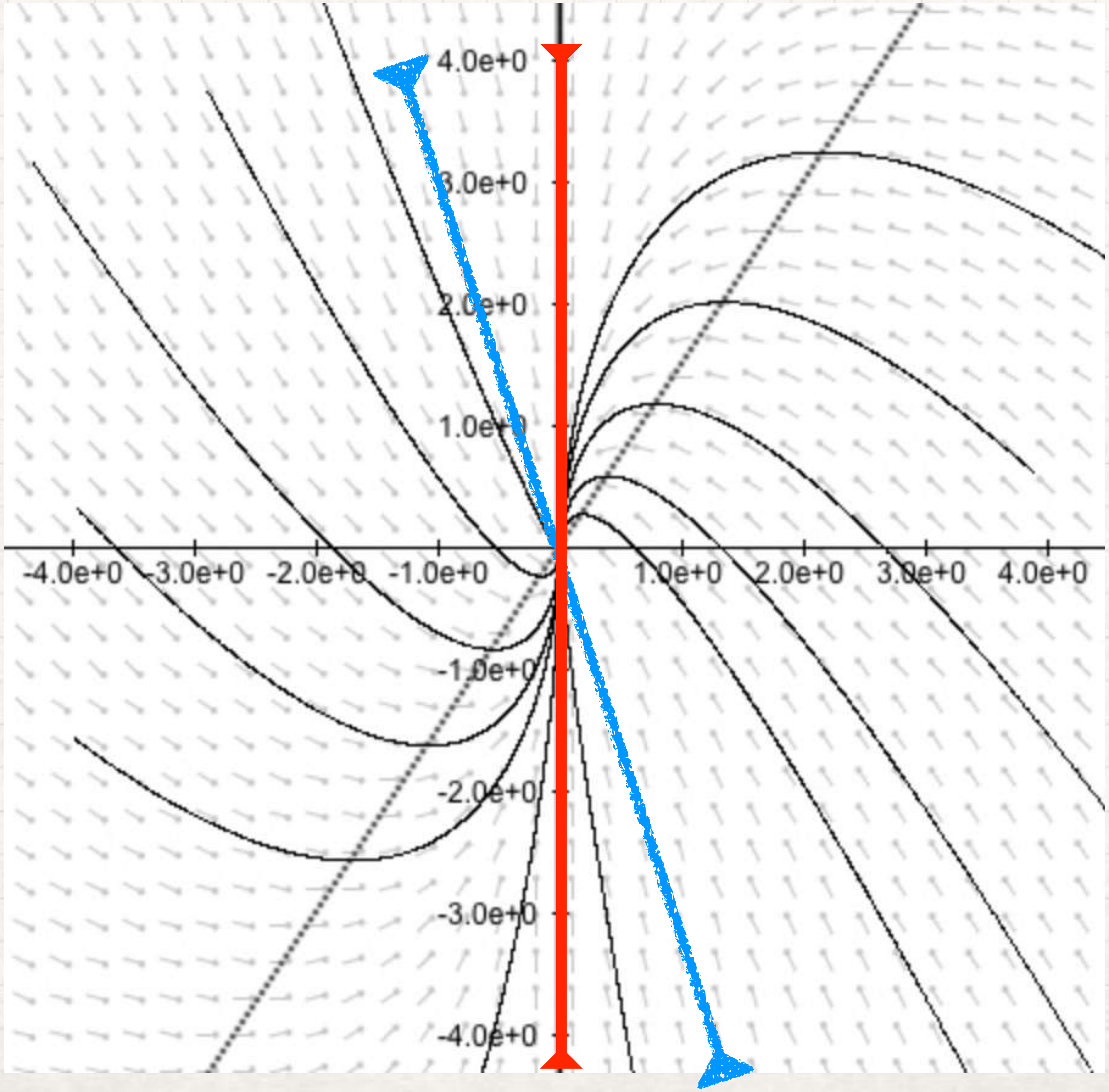
$$\lambda_2 = -3, \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$



Phase Plane Analysis of Linear systems

$$\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$$

Note that the vector field starts parallel to V1 and ends parallel to V2 .



Phase Plane Analysis of Linear systems

Example 3: Draw the Phase portrait of the following system.

$$\dot{x} = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix} x$$

Sol.

1.The equilibrium point is $x_{eq} = (0,0)$

2.The eigenvalues and eigenvectors are $\lambda_1 = 4,\ \vec{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $,\lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.The equilibrium point is a Saddle node.

4.Draw the eigenvectors and nullclines on the phase plane.

5.Now, take few points on the eigenvectors above and below each and check the direction.

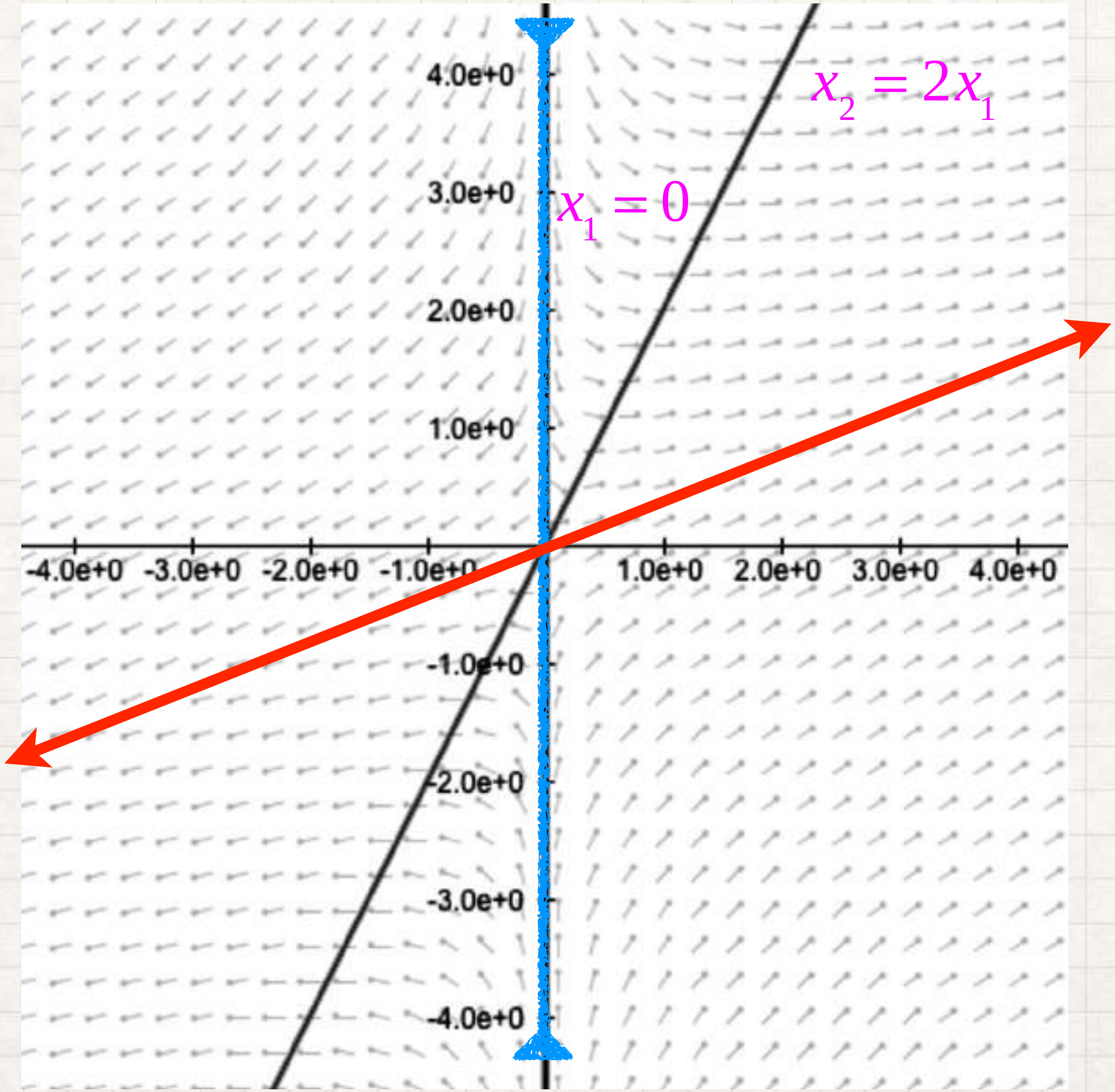
Concepts of Phase Plane Analysis

Example 3:

$$\dot{x} = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix} x$$

$$\lambda_1 = 4, \ \ \vec{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



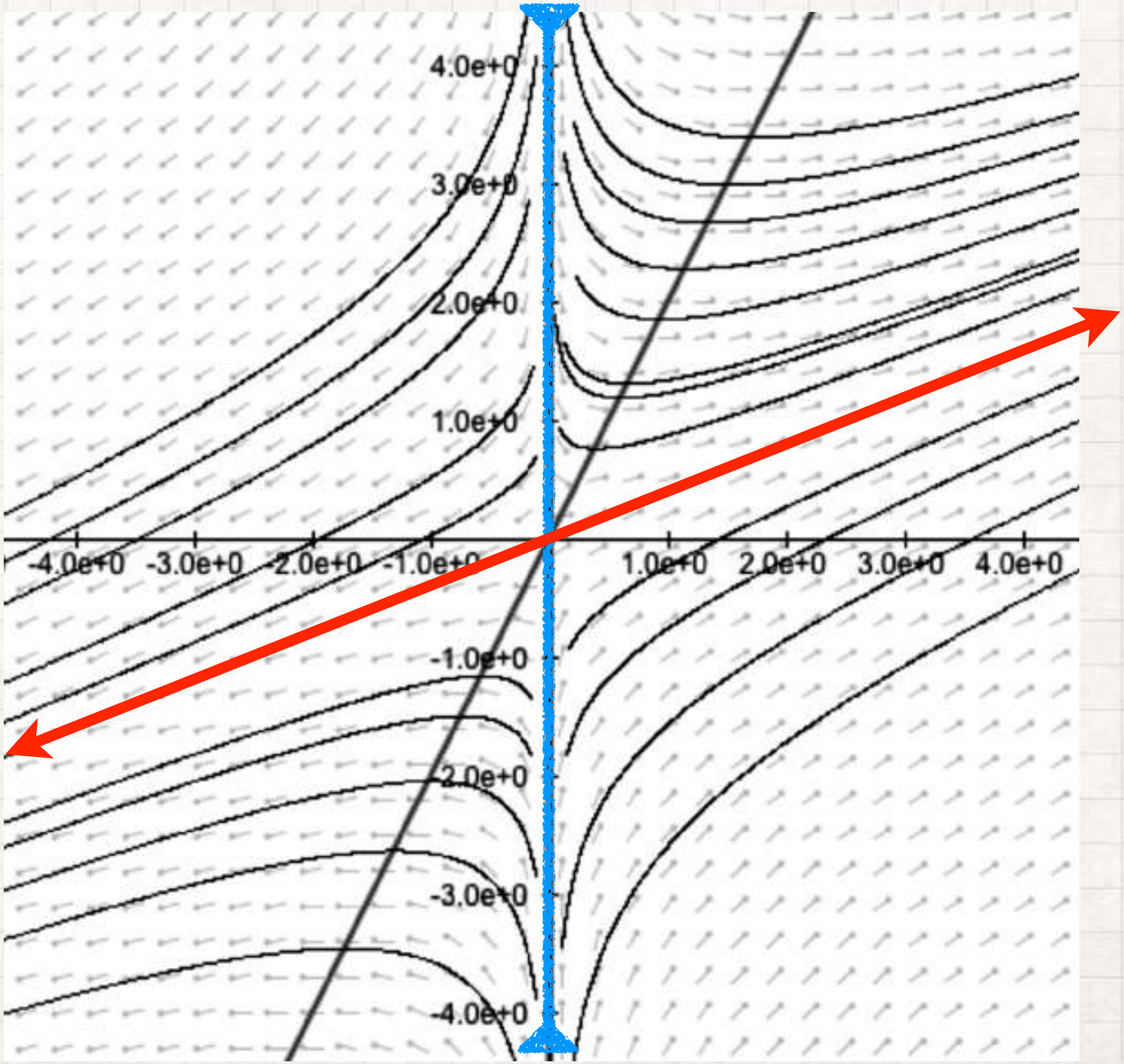
Phase Plane Analysis of Linear systems

Example 3:

$$\dot{x} = \begin{pmatrix} 4 & 0 \\ 2 & -1 \end{pmatrix} x$$

$$\lambda_1 = 4, \quad \vec{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Concepts of Phase Plane Analysis

Example 4: Draw the Phase portrait of the following system.

$$\dot{x} = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix} x$$

Sol.

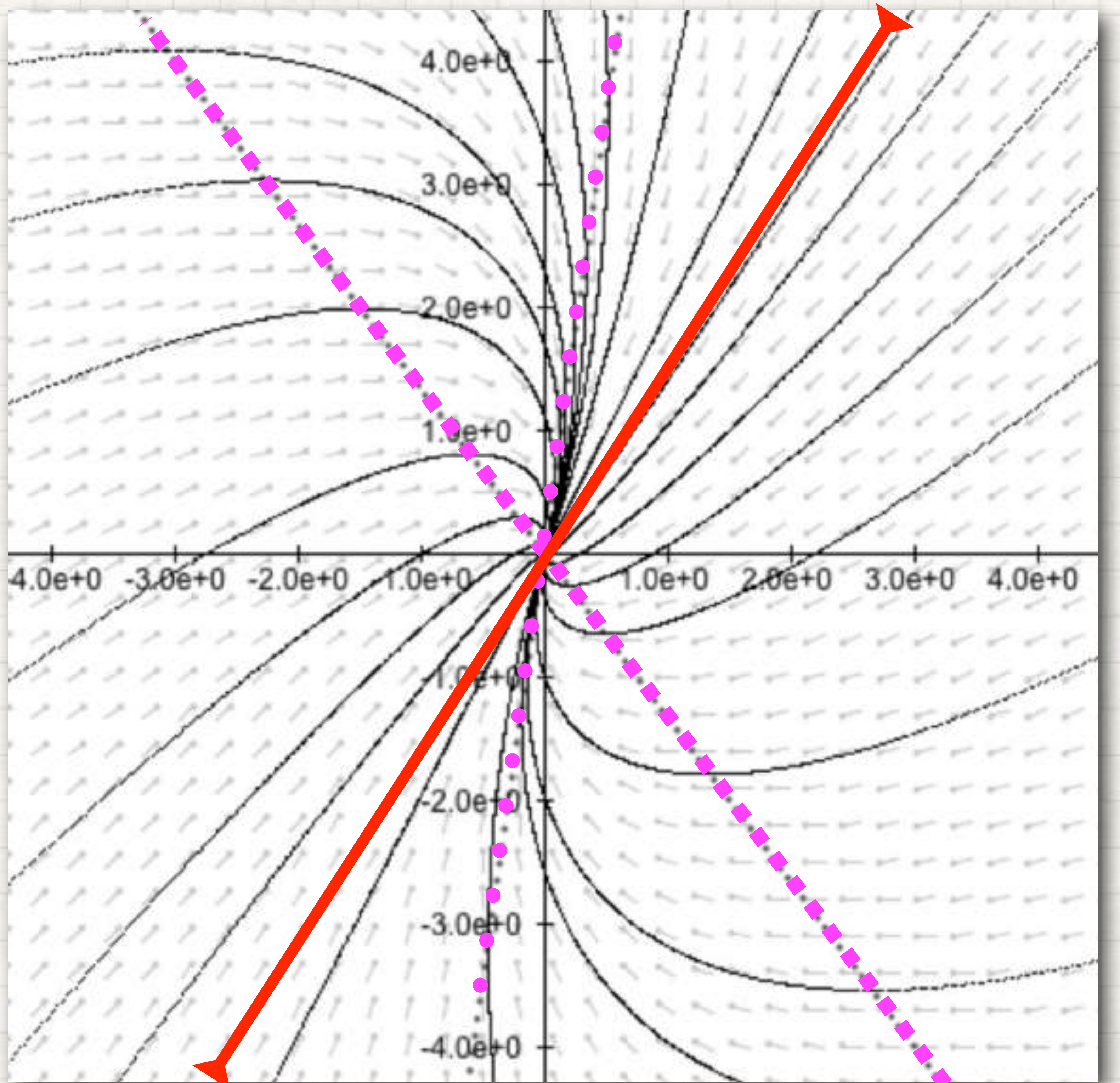
1. The equilibrium point is $x_{eq} = (0,0)$
2. The eigenvalues and eigenvectors are $\lambda_{1,2} = -5, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
3. The equilibrium point is a <u>Stable node.</u>
4. <u>Draw the eigenvector and nullclines on the phase plane.</u>
5. <u>Now, take few points on the eigenvectors above and below each and check the direction.</u>

Phase Plane Analysis of Linear systems

Example 4:

$$\dot{x} = \begin{pmatrix} -7 & 1 \\ -4 & -3 \end{pmatrix} x$$

$$\lambda_{1,2} = -5, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



HOME WORKS

- Draw phase portrait of the following systems:

1. $\dot{x} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix} x$ 2. $\dot{x} = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix} x$ 3. $\dot{x} = \begin{pmatrix} 2 & 3 \\ -3 & -2 \end{pmatrix} x$

2.Diagonalize (put in Jordan form) the following system matrices (if possible) and draw the phase portrait for the original and uncoupled system (see section 2.1 of [4]):

$$\dot{x} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} x, \qquad \dot{x} = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} x$$

4.Prove that for a system having the solution vector $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \quad for \quad \lambda_1 \neq \lambda_2$, for $c_1 = 0$ the resulting trajectory will be on the eigenvector v_2.

5.Prove graphically that in the phase plane for $\dot{x} = \begin{pmatrix} -3 & 0 \\ 3 & -2 \end{pmatrix} x$ if the initial point xo is on any of the eigenvectors, the resulting trajectory remains on that eigenvector.

Why an Equilibrium point is called Singular Point ? [1]

- To answer this, let us examine the slope of the phase trajectories.

- The slope of the phase trajectory passing through a point (x_1,x_2) is determined by

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

- With the functions f_1 and f_2 assumed to be single valued, there is usually *a definite value for this slope* at any given point in phase plane. This implies that the *phase trajectories will not intersect.*

- *At singular points, however,* ***the value of the slope is 0/0****, i.e., the slope is* ***indeterminate****. Many trajectories may intersect at such points.*

- This indeterminacy of the slope accounts for the adjective "*singular*".

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