Lect. 1 Control and system Engineering Department
Mathematics III / Third year

(*) Series Solution of Differential Equations:-

(#) power series Method :-

Def: The power series method is the standard basic method for solving Linear differential equations with variable coefficient (Function of x), These Series can be used for Computing Values, graphing curve, proving formulas etc.

power Series (in powers of $x - x_0$)

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

$a_0, a_1, a_2, \cdots$ are Constant and x_0 is a constant called the center of the series.

power Series (in $x_0 = 0$)

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$$

Examples of Power Series (Maclaurin Series):-

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad , (|x| < 1) \text{ (geometric series)}$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum^{\infty} \frac{(-1)^m x^{2m}}{} \quad x^2 \quad x^4$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + - - - -$$

Idea and Technique of the power Series Method:-

Consider the following DE:-

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad (1)$$

the idea of Power Series Solution Method is explained by the following steps:

1st Step represent $p(x)$ and $q(x)$ by Powers of x OR $(x - x_0)$.

$$\left.\begin{aligned} p(x) &= P_0 + P_1 x + P_2 x^2 + - - - - - \\ q(x) &= q_0 + q_1 x + q_2 x^2 + - - - - - \end{aligned}\right\} \qquad (2)$$

2nd step assume a solution in the form of a power Series with unknown Cofficient

$$y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + - - - - \qquad (3)$$

also the 1st and 2nd derivatives of y:

$$y'(x) = \sum_{m=0}^{\infty} m\, a_m x^{m-1} = \sum_{m=1}^{\infty} m\, a_m x^{m-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + - - - - - \qquad (4)$$

$$y''(x) = \sum_{m=0}^{\infty} m(m-1)\, a_m x^{m-2} = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2}$$

$$= 2a_2 + 6a_3 x + 12 a_4 x^2 + - - - - \qquad (5)$$

3rd Step Sub. eqs. (2) $\rightarrow$ (5) in eq. (1) and Collect Like power of x and equation the Sum of the cofficient of each occuring power of x to Zero. This gives relations from which we can

Problem Set (A) :- Apply the power Series method To solve the following differential equations: $X_0=0$.

A.1 $y' + 2y = 0 \rightsquigarrow$ ans.) $y = ce^{-2x}$

A.2 $y' = 2xy \rightsquigarrow$ ans.) $y = ce^{x^2}$

A.3 $y'' + 9y = 0 \rightsquigarrow$ ans.) $y = c_1 \cos 3x + c_2 \sin 3x$

A.4 $y' = 3x^2 y$

A.5 $y'' = y'$ $\rightsquigarrow$ ans.) $y = c_1 + c_2 e^{x}$, $c_1 = a_0$

A.6 $(1-x)y' = y \rightsquigarrow$ ans.) $y = \frac{a_0}{1-x}$

Lect.2 Theory of Power Series Method :-

* Operation on Power Series :-

[a] Termwise Differentiation.

A power Series may be differentiated term by term.

More precisely if

$$y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$$

Then $\rightsquigarrow$ $$y'(x) = \sum_{m=0}^{\infty} m\, a_m (x-x_0)^{m-1} = \sum_{m=1}^{\infty} m\, a_m (x-x_0)^{m-1}$$

and $\rightsquigarrow$ $$y''(x) = \sum_{m=0}^{\infty} m(m-1)\, a_m (x-x_0)^{m-2}$$

$$= \sum_{m=0}^{\infty} m(m-1)\, a_m (x-x_0)^{m-2}$$

b Termwise Addition:-

Two power Series may be added term by term as:

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m + \sum_{m=0}^{\infty} b_m (x-x_0)^m = \sum_{m=0}^{\infty} (a_m+b_m)(x-x_0)^m$$

c Shifting Summation Indices :-

This is best explained in terms of a typical example Like in the following:

$$x^2 \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$\rightarrow \sum_{m=2}^{\infty} m(m-1)\, a_m x^m + \sum_{m=1}^{\infty} m\, a_m x^{m-1}$$

1st step shift the power of x in each term to the highest power, i.e.:-

Let $s=m-1$. for the second term $\rightsquigarrow$ $m=s+1$

then we obtain:

$$\sum = \sum_{m=2}^{\infty} m(m-1)\, a_m x^m + \sum_{s=0}^{\infty} (s+1) a_{s+1} x^s$$

2nd Step start the summation in each term from Zero :-

$$\sum = \sum_{s=0}^{\infty} s(s-1)\, a_s x^s + \sum_{s=0}^{\infty} (s+1) a_{s+1} x^s$$

$$= \sum_{s=0}^{\infty} \left[s(s-1)\, a_s + (s+1)\, a_{s+1} \right] x^s$$

$$= a_1 + 2a_2 x + (2a_2 + 3a_3) x^2 + \cdots$$

Note that :- $\sum_{m=2}^{\infty} m(m-1)\, a_m x^m = \sum_{m=0}^{\infty} m(m-1)\, a_m x^m = \sum_{s=0}^{\infty} s(s-1)\, a_s x^s$

Legendre's Equation
Legendre polynomial $P_n(x)$

Legendre Equation :-

Legendre equation is given by:

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{----} \quad (10)$$

The parameter (n) in (10) is a given real number and any solution of (10) is called Legendre function.

NOTE :-

The $p(x)$ and $q(x)$ for equation (10) is analytic at $x=0$ (but not at $x=\pm 1$). therefore an analytic power Series Solution exist with a radius of Convergence not Less than one. i.e. $1<|x|<1$

To derive the power series Solution for the Legendre equation Subsititute:-

$$y = \sum_{m=0}^{\infty} a_m x^m$$

and its derivation into (10) and denoting $n(n+1)$ by k, we obtain,

$$(1-x^2)\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=0}^{\infty} m\,a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2\sum_{m=0}^{\infty} m\,a_m x^m + k\sum_{m=0}^{\infty} a_m x^m = 0$$

The first term :-

Let $m-2 = s$

$m = s+2 \quad , \quad m=0$

$s=-2$

$$\sum^{\infty} (s+2)(s+1)a\ x^s - \sum^{\infty} s(s-1)a_s x^s - 2\sum^{\infty} s\,a_s x^s$$

$$\rightsquigarrow \sum_{s=0}^{\infty}(s+2)(s+1)\,a_{s+2}\,x^{s} - \sum_{s=0}^{\infty} s(s-1)\,a_s\,x^{s} - 2\sum_{s=0}^{\infty} s\,a_s\,x^{s} + k\sum_{s=0}^{\infty} a_s\,x^{s} = 0$$

$$\rightsquigarrow \sum_{s=0}^{\infty}\Big[(s+2)(s+1)\,a_{s+2} + [s(s-1) - 2s + k]\,a_s\Big]x^{s} = 0$$

$$\rightsquigarrow \sum_{s=0}^{\infty}\Big[(s+2)(s+1)\,a_{s+2} + (k - s^2 - s)\,a_s\Big]x^{s} = 0$$

$$a_{s+2} = -\frac{k - s^2 - s}{(s+2)(s+1)}\,a_s \quad , \; s = 0, 1, 2, \ldots \qquad ---- (11)$$

The term $(k - s^2 - s)$ may be written as:

$$k - s^2 - s = n(n+1) - s^2 - s$$

$$n(n+1) - s(s+1) = n^2 + n - s^2 - s$$

$$(n^2 - s^2) + (n - s) = (n-s)(n+s) + (n-s)$$

$$= (n-s)[n+s+1]$$

Then (11) becomes:-

$$a_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\,a_s \quad , \; s = 0, 1, 2, \ldots$$

$$---- (12)$$

Relation (12) Called a recurrence relation or recursion formula.

For (12), a_0 and a_1 are arbitrary Constant

The other parameters are :

$s=0 \rightarrow \boxed{a_2 = -\frac{n(n+1)}{2!} a_0}$

$s=1 \rightarrow \boxed{a_3 = -\frac{(n-1)(n+2)}{3!} a_1}$

$s=2 \rightarrow a_4 = \frac{-(n-2)(n+3)}{4*3} a_2$

$$= \frac{-(n-2)(n+3)}{4*3} * \frac{-n(n+1)}{2!} a_0$$

$$\boxed{a_4 = \frac{(n-2)\, n\, (n+1)(n+3)}{4!} a_0}$$

$s=3 \rightarrow a_5 = -\frac{(n-3)(n+4)}{5*4} a_3$

$$= \frac{-(n-3)(n+4)}{5*4} * \frac{-(n-1)(n+2)}{3!} a_1$$

$$\boxed{a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1}$$

Then we obtain the following series solution

$$y = a_0 y_1(x) + a_1 y_2(x) \qquad \text{-----} \; (13)$$

where,

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)\, n\, (n+1)(n+3)}{4!} x^4 - + \cdots \qquad \text{---} \; (14)$$

and

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - + \cdots \qquad \text{---} \; (15)$$

These Series Converges for $|x|<1$ and the two Solution y_1 and y_2 are linearly independent Solutions.

Lect. 4

Frobenius Method

Power Series Solution about a Singular Points.

In the preceding Lectures we see that two Linearly independent Solutions exist for the following 2^{nd} order differential equation.

$$A(x)\, y'' + B(x)\, y' + C(x)\, y = 0$$

about an ordinary point $x = x_0$. However when $x = x_0$ is a singular point, it is not always possible to find a solution in the form $y = \sum_{m=0}^{\infty} a_m (x - x_0)^m$.

Regular and Irregular Singular Points:-

Singular points are classified as either regular or irregular singular points.

Def.:- Regular and Irregular Singular points.
A singular point $x = x_0$ of equation (1) is said to be a regular singular point if both $(x - x_0)\, p(x)$ and $(x - x_0)^2\, Q(x)$ are analytic at x_0.

A singular point not regular at x_0 is said to be an irregular singular point.

Indicial Equation:-

In general,

if $x=0$ is a regular singular point of (1), then the function $xP(x)$ and $x^2Q(x)$ are analytic, i.e.,

$$xP(x) = P_0 + P_1 x + P_2 x^2 + \cdots$$
$$x^2 Q(x) = q_0 + q_1 x + q_2 x^2 + \cdots$$

After Substituting $y = \sum_{m=0}^{\infty} a_m x^{m+r}$ and its derivatives in (4), Simplifying and equating the total cofficients of the lower power of x to Zero, we obtain the Indicial equation as:

$$\boxed{r(r-1) + P_0 r + q_0 = 0} \quad \text{-----} \quad (21)$$

The root of (21) are r_1 and r_2.

Cases of Indicial Roots :-

Case I : Roots not differing by an Integer.

$\boxed{r_1 - r_2 \neq N}$, <u>N is integer $(r_1 > r_2)$</u>, Then the solution are :-

$$y_1(x) = x^{r_1} \sum_{m=0}^{\infty} a_m x^m = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$y_2(x) = x^{r_2} \sum_{m=0}^{\infty} b_m x^m = x^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots)$$

$$\text{-----} \quad (22)$$

Lect. 5

Bessel's Equation

One of the most important differential equation in applied Mathematics is Bessel's differential equation.

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0} \qquad \text{— (1)}$$

Bessel's equation can be solved by the Frobenius Method where $x_0 = 0$ is a regular singular point.

Accordingly, we have

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad \text{— (2)}$$

Substitute eq. (2) and it's derviatives in eq. (1), we obtain

$$\rightsquigarrow x^2 \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} + x \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + (x^2 - \nu^2) \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\rightsquigarrow \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r) a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\rightsquigarrow \sum_{m=0}^{\infty} \left[(m+r)^2 - \nu^2 \right] a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$$

$$\rightsquigarrow x^r \left\{ \underbrace{\sum_{m=0}^{\infty} \left[(m+r)^2 - \nu^2 \right] a_m x^{\overset{s+2}{m}}}_{\substack{m = s+2 \\ m=0,\ s=-2}} + \underbrace{\sum_{m=0}^{\infty} a_m x^{\overset{s+2}{m+2}}}_{\substack{m=s \\ m=0,\ s=0}} \right\} = 0$$

$$\rightsquigarrow x^r \left\{ \sum_{s=-2}^{\infty} \left[(s+r+2)^2 - \nu^2 \right] a_{s+2} x^{s+2} + \sum_{s=0}^{\infty} a_s x^{s+2} \right\} = 0$$

$$x^r \left\{ \left[(r)^2 - \nu^2\right] a_0 x^0 + \left[(r+1)^2 - \nu^2\right] a_1 x + \sum_{s=0}^{\infty} \left(\left[(s+r+2)^2 - \nu^2\right] a_{s+2} + a_s \right) x^{s+2} \right\} = 0$$

The indicial Equation :

$$r^2 - \nu^2 = 0 \quad , \; a_0 \neq 0 \qquad \text{— (3)}$$

The roots are, $r_1 = \nu_1$ and $r_2 = -\nu_2$. Also $a_1 = 0$

The recurrence equation is given by:

$$a_{s+2} = \frac{-1}{(s+r+2)^2 - \nu^2} a_s \quad , \; s = 0, 1, 2, \ldots \qquad \text{— (4)}$$

First recurrence equation in the case of $r = r_1 = \nu$

$$a_{s+2} = \frac{-1}{(s+\nu+2)^2 - \nu^2} a_s$$

$$a_{s+2} = \frac{-1}{(s+2)(s+2+2\nu)} a_s \qquad \text{— (5)}$$

Hint :-

$(s+\nu+2)^2 - \nu^2$,

$[(s+2)+\nu]^2 - \nu^2$

$(s+2)^2 + 2\nu(s+2) + \cancel{\nu^2} \cancel{-\nu^2}$

$(s+2)^2 + (s+2) * 2\nu$

$(s+2)[s+2+2\nu]$

s=0

$$a_2 = \frac{-1}{2(2+2\nu)} a_0$$

$$a_2 = \frac{-1}{2^2(1+\nu)} a_0$$

s=1

$$a_3 = 0$$

s=2

$$a_4 = \frac{-1}{2^2(2\nu+4)} a_2$$

$$= \frac{-1}{2^3(\nu+2)} a_2 = \frac{1}{2^4 \cdot 2!(\nu+1)(\nu+2)} a_0$$

(*) Bessel Function of the Second Kind $Y_\nu(x)$:-

For noninteger ν we already have a basis J_ν and $J_{-\nu}$ (two independent solution), but for $\nu = n$ these two solutions becomes Linearly dependent, so we need a second independent Solution, This solution will be denoted by $Y_n(x)$.

Bessel Function of the Second kind $Y_n(x)$:-

For $\nu = n = 1, 2, 3, \ldots$ a second solution may be put in the following form

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\left[J_\nu(x) \cos \nu\pi x - J_{-\nu}(x) \right] \quad \text{--- (21.a)}$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x) \quad \text{--- (21.b)}$$

This Function is called the Bessel function of the second kind of order ν.

Remarks:-

For noninteger ν, the function $Y_\nu(x)$ is evidently a solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are solution of that equation.

For ν equal to integer, then the Limit in (21.b) exist and $Y_n(x)$ is a second Solution of the Bessel equation.

The Form of $Y_n(x)$ is given by the following Formula.

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\left(\frac{x}{2}\right) + \gamma\right) + \frac{x}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n}\cdot m!\cdot (m+n)!} x^{2m}$$

$$- \frac{x^{-n}}{\pi}\sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n}\cdot m!} x^{2m} \quad \text{--- (22)}$$

where $x > 0$, $n = 0, 1, \ldots$, and

$h_0 \neq 0$, $h_s = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}$ $(s = 1, 2, \ldots)$

and $(\gamma) = 0.57721566 \ldots\ldots$ is Euler's constant

Theorem (4): (General solution of Bessels' equation).

A general solution of Bessels' equation for all value of ν is:

$$\boxed{y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)} \quad \text{—— (23)}$$

Problem Set (F) :-

In this problem, use the indicated substitutions, reduce the following equation to Bessels' differential equation and then find the general solution in terms of Bessel function.

F.1: $4x^2 y'' + 4xy' + (100x^2 - 9)y = 0$ $\quad (5x = z)$

Ans: $y = c_1 J_{\frac{3}{2}}(5x) + c_2 Y_{\frac{3}{2}}(5x)$

F.2: $y'' + k^2 xy = 0$ $\quad (y = u\sqrt{x},\ \frac{2}{3}kx^{\frac{3}{2}} = z)$

F.3: $y'' + k^2 x^2 y = 0$ $\quad (y = u\sqrt{x},\ \frac{1}{2}kx^2 = z)$

Ans: $\sqrt{x}\left(c_1 J_{\frac{1}{4}}(\frac{1}{2}kx^2) + c_2 Y_{\frac{1}{4}}(\frac{1}{2}kx^2)\right)$

F.4: $x^2 y'' + (1 - 2\nu)xy' + \nu^2(x^{2\nu} + 1 - \nu^2)y = 0$

$(y = x^\nu u,\ x^\nu = z)$

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Assest. Lect. Rua'a Muayad

Complex Number

$\mathbb{R}/\mathbb{m}$

Introduction:-

Def: A complex number z is an ordered pair $z(x,y)$ of real number x and y, written.

$z = (x,y)$

x is called the real part and y the imaginary part of z, written

$x = \operatorname{Re} z \quad , \quad y = \operatorname{Im} z$

or, in a complex notation.

$$\boxed{z = x + iy}$$

where

$i^2 = -1 \quad , \quad i = \sqrt{-1}$

* **Addition:** of two complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

* **Multiplication:** of z_1 and z_2 $(z_1 z_2)$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

* **Subtraction:** $(z_1 - z_2)$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

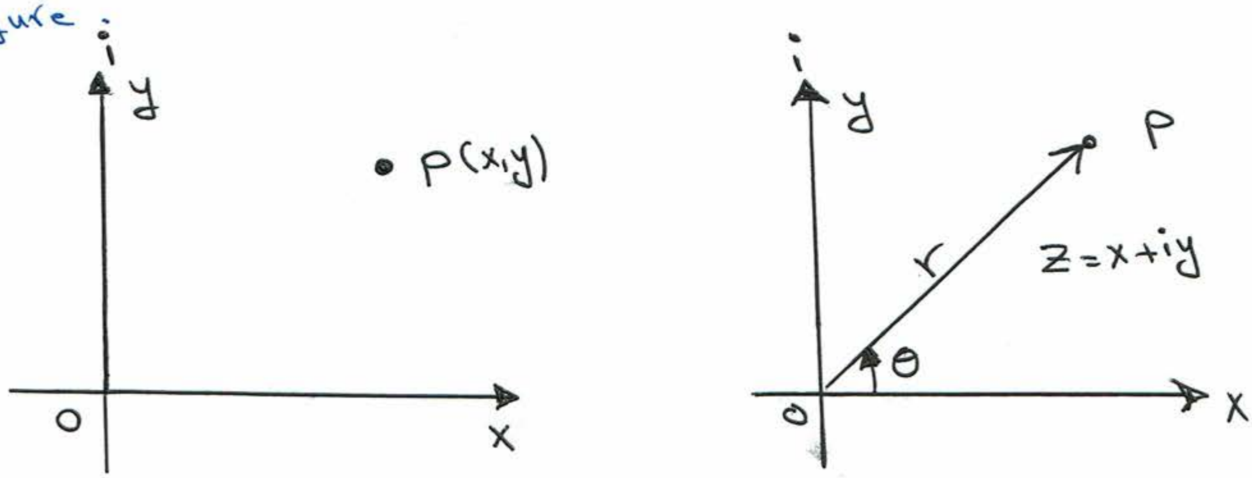
* **Division:** (z_1 / z_2) $(z_2 \neq 0)$

$$z = \frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)$$

NOTE:- To get the result of division (z_1/z_2), we multiply numerator and denominator of (z_1/z_2) by $\bar{z}_2$ (i.e. $x_2 - iy_2$)

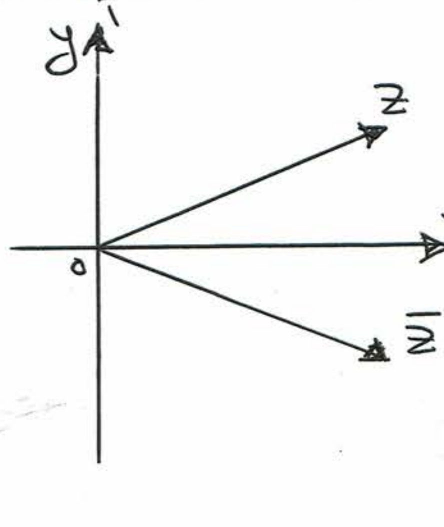
* Complex plane:-

In the complex plane, (x,y) plane, the complex number z viewed as a point (p) in this plane of ordered pair (x,y) or as a vector (op) as shown in the following Figure:



Complex conjugate numbers:-

The complex conjugate $\bar{z}$ of a complex number $z = x+iy$ is defined as:



$$\boxed{\bar{z} = x - iy}$$

therefore x and y written as

$$\boxed{x = \frac{z+\bar{z}}{2}} \quad , \quad \boxed{y = \frac{z-\bar{z}}{2i}}$$

Also, we have

$*\ \overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$, $*\ \overline{(z_1 z_2)} = \bar{z}_1\, \bar{z}_2$

and $*\ \overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

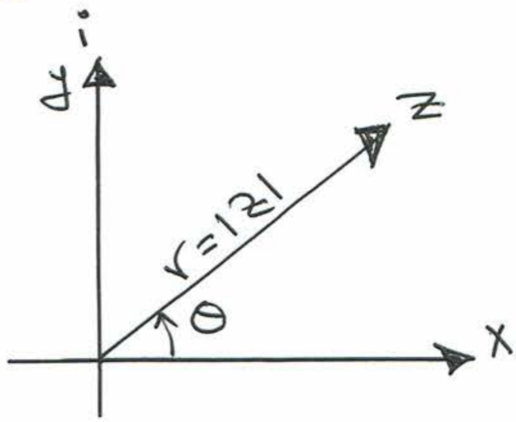
Lect.8

polar Form of Complex Number Z :-

The x and y Components of the Complex number Z, may also represented by the usual polar Coordinates (r, θ) defined by.

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

Then Z in polar form



$$\boxed{Z = r(\cos\theta + i\sin\theta)}$$

where

$$\boxed{r = \sqrt{x^2+y^2} = |Z| = \sqrt{Z\bar{Z}}}$$

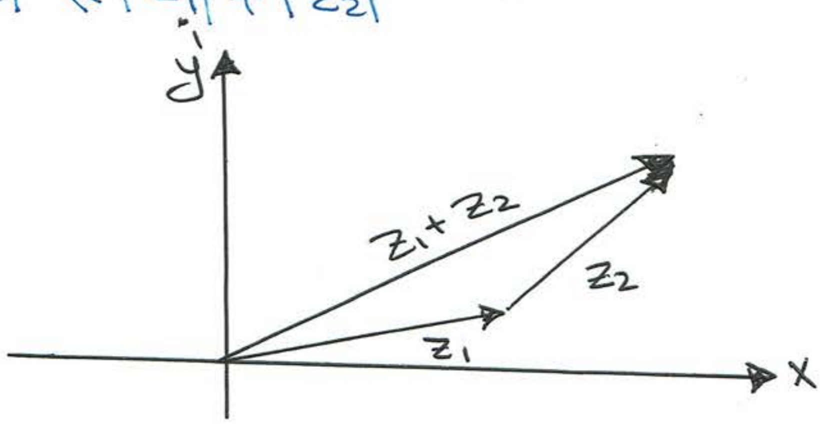
$$\boxed{\theta = \arg Z = \tan^{-1}\left(\frac{y}{x}\right)}$$

The angle θ by definition Satisfy the following

$$-\pi < \arg Z \leq \pi$$

Triangle Inequality :- For any complex numbers we have the important triangle inequality

$$|Z_1 + Z_2| \leq |Z_1| + |Z_2|$$



Multiplication and Division in polar Form :-

The multiplication of Z_1 and Z_2 in polar form is given by

$$\boxed{Z_1 Z_2 = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2) \right]}$$

where : $r_1 = |Z_1|$, $r_2 = |Z_2|$ and $|Z_1 Z_2| = |Z_1||Z_2| = r_1 r_2$

also: $\arg(Z_1 Z_2) = \arg Z_1 + \arg Z_2$

More General Formula:

$$\boxed{Z_1 Z_2 Z_3 \cdots Z_n = r_1 r_2 r_3 \cdots r_n \left[\cos(\theta_1+\theta_2+\theta_3+\cdots+\theta_n) + i \sin(\theta_1+\theta_2+\theta_3+\cdots \theta_n) \right]}$$

LeT $Z_1 = Z_2 = Z_3 = \cdots = Z_n = r(\cos\theta + i \sin\theta)$

we obtain

$$\boxed{Z^n = r^n (\cos n\theta + i \sin n\theta) \quad , n = 1, 2, 3, \ldots} \quad (1)$$

when $r=1$, formula (1) becomes De Moiver's Formula:

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta \quad (2)$$

Roots:-

if $Z = W^n$, $n = 1, 2, 3, \ldots$ oR $W = (Z)^{\frac{1}{n}}$

then the n^{th} root of Z is given by

$$\boxed{W = \sqrt[n]{Z} = \sqrt[n]{r} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) \pm i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right)} \quad (3)$$

where $k = 0, 1, 2, \ldots, n-1$. when $k=0$, the value of $\sqrt[n]{Z}$ is the principle value of W

Lect.9

Function of Complex Variable :-

if $z = x + iy$ and $w = u + iv$ are two Complex Variable then w is said to be a function of z as

$$W = f(z)$$
$$= u(x,y) + i v(x,y)$$

Example (3) :- Let $w = f(z) = z^2 + 3z$. Find u and v

Sol :-

$$f(z) = (x+iy)^2 + 3(x+iy)$$
$$= x^2 + i2xy - y^2 + 3x + i3y$$
$$= \underbrace{(x^2 - y^2 + 3x)} + i\underbrace{(2xy + 3y)}$$
$$= u + iv$$

Example (4) :- Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v

Sol:-

$$w = f(z) = 2iz + 6\bar{z}$$
$$= 2i(x+iy) + 6(x-iy)$$
$$= i2x - 2y + 6x - i6y$$
$$= \underbrace{(6x-2y)}_{u} + i\underbrace{(2x-6y)}_{v}$$

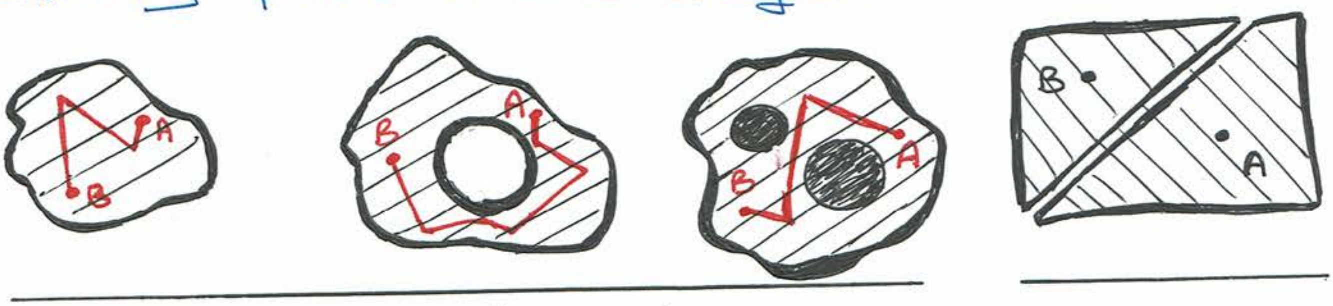
where, $\boxed{u = \operatorname{Re} W}$ and $\boxed{v = \operatorname{Im} W}$

Hint :- Function defined here as a single-valued function (one to one function).

Concepts on Sets in the Complex plane.

A point set (S) in the Complex plane means any sort of collection of finitly many or infinitely many point.

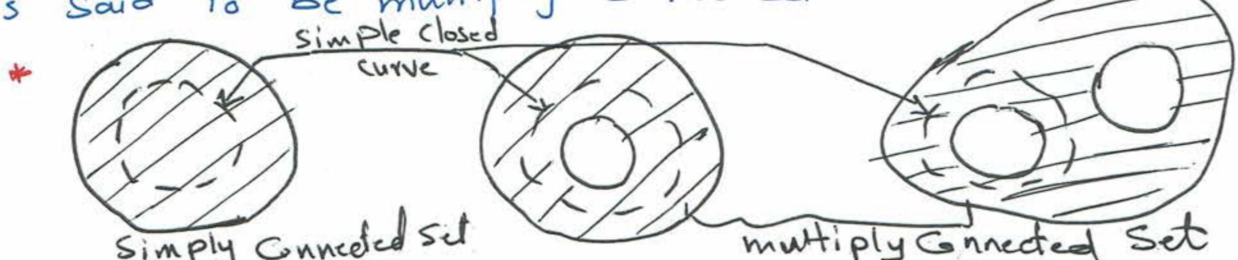
An open Set (S) (Like $|z|<1$) is Called Connected if any two of it's points can be joined by a broken Line of finitely many straight Line segments all of whose points belong to S. An open Connected Set is called a Domain. A set consisting of a domain with non, some, or all its boundary points is called a Region.



Connected Set

Disconnected Set

A connected Set S with the property that every simple* closed curve. which can be drawn in its interior Continous only points of S is said to be Simply Connected. If it's possible to draw in S at Least one simple closed curve whose interior Continous one or more points not belonging to S when S is said to be multiply Connected.



Complex Number

Assist. Lect:
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Analytic Function:-

Definition (Analyticity):- A Function $f(z)$ is said to be analytic in a domain D, if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z=z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Cauchy-Riemann Equation. Laplace's Equation.

The Cauchy-Rieman equation provide a criterion (a test) for the analyticity of a complex function

$$\boxed{W=f(z)=u(x,y)+i\,v(x,y)}$$

Roughly, $f(z)$ is analytic in a domain D if and only if it satisfy the two so-called Cauchy-Riemann equation

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\quad,\quad\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}} \qquad \text{--- (8)}$$

Hence, if $f(z)$ is analytic in a domain D, those partial drivatives exist and satisfy (8) at all points of D.

Remark:- Cauchy-Riemann equation are not only necessary but also sufficient conditions for the existance of the derivative of $f(z)=W=u+iv$ at $z=z_0$. In this case the derivative of $f(z)$ is given by:

$$\boxed{f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\quad \text{or} \quad f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}}$$

$$\text{--- (9)}$$

Example _ (9) :- Is $f(z) = z^3$ analytic ?

Sol:- $f(z) = (x+iy)^3 = u + iv$

where u and v are found as

$$f(z) = (x+iy)(x+iy)^2$$

$$= (x+iy)(x^2 + i2xy - y^2)$$

$$= x^3 + i2x^2y - xy^2 + ix^2y - 2xy^2 - iy^3$$

$$= (x^3 - xy^2 - 2xy^2) + i(2x^2y + x^2y - y^3)$$

$$= \underbrace{(x^3 - 3xy^2)}_{u} + i\underbrace{(3x^2y - y^3)}_{v}$$

Applying Cauchy - Riemann equations:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad , \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

&

$$\frac{\partial u}{\partial y} = -6xy \quad , \quad -\frac{\partial v}{\partial x} = -6xy$$

$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore f(z) = z^3$ is an analytic function.

Example _ (10) :- Find the derivative of $f(z) = z^3$

Sol:-
Since $f(z) = z^3$ is analytic function (From example _(9))

then , we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3x^2 - 3y^2 + i(6xy)$$

OR

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 3x^2 - 3y^2 + i(6xy)$$