Digital signals Processing (DSP)
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Chapter One

Introduction

1.1 Basic Elements of DSP Systems:

Fig. (1.1) shows a typical DSP system. The analog input Signal might be variations in voltage, temperature pressure or light intensity. If the signal is not inherently electrical, it is first converted to a proportional voltage fluctuation by a suitable transducer. Very often, the first stage in the chain is an analog filter , designed to limit the frequency range of the signal prior to sampling. The final processing stage is another analog filter, designed to remove sharp transitions from the D/A output.

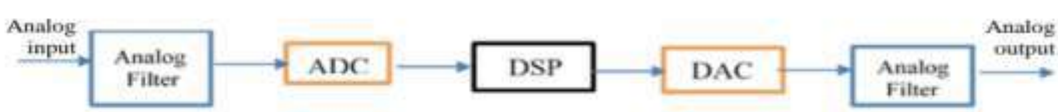


Figure1.1:Typical DSP scheme

In contrast to the above a direct analog processing of analog signals is much simpler since it involves only a signal processor. It is therefore natural to ask why do we go to the DSP system? There are several good reasons:

1- Rapid advances in integrated circuit design and manufacture are producing more powerful DSP systems on a single chip at decreasing size and cost.

2- Digital processing is inherently stable and reliable.

3- In many cases DSP is used to process a number of signals simultaneously. This may be done by using a technique known as " TDM " time-division-multiplexing

4 - Digital implementation permits easy adjustment of processor characteristics during processing, such as that needed in implementing adaptive circuits.

Today, digital signal processing techniques are increasingly replaced analog signal processing methods in many fields such as spectral analysis, speech recognition, radar and sonar signal processing, biomedical signal analysis, digital filtering, digital modems, data encryption, geophysical signal processing and engine control.

☆☆The main disadvantage associated with DSP is the limited range of frequencies available for processing.

1.2 Sampling and A/D Conversion: -

The sampling process is confirmed by **Shannon's famous Sampling theorem** which may be stated as follows :

An analog signal containing frequencies up to (fm) Hz completely represented by regularly-spaced samples, provided the sampling rate is at least (2 fm) samples per second i.e **(fs > 2 fm).**

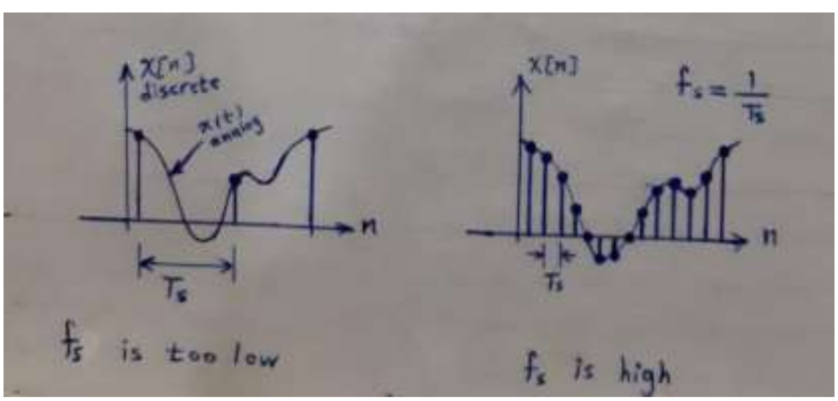


Figure1.2: sampled signal

Fig (1.3) shows the effects of sampling signal spectrum.

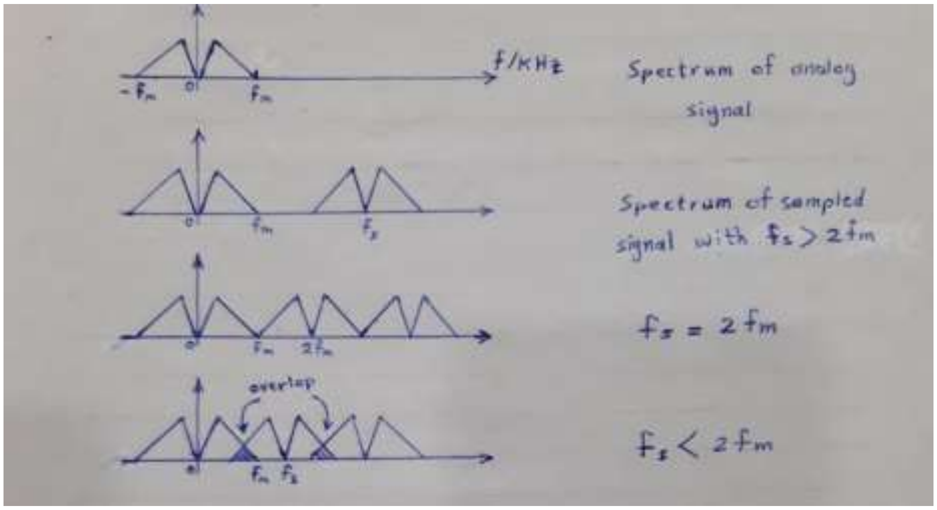
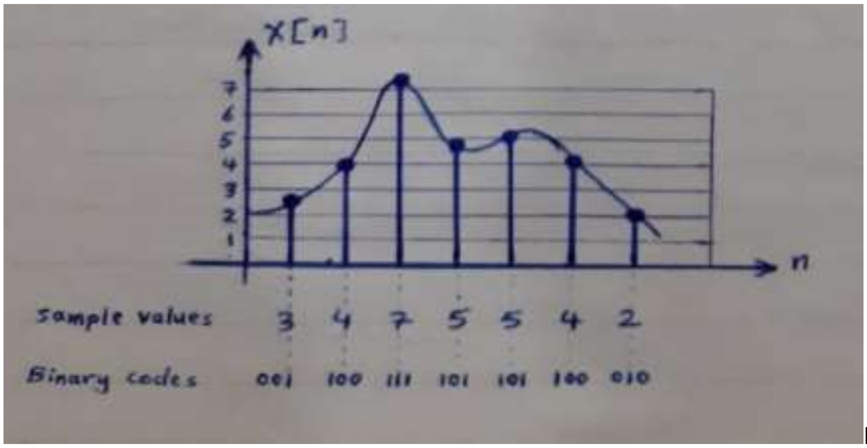


Fig (1.3)

The maximum frequency contained in the analog signal (fm) is known as **Nyquist frequency**. The minimum sampling rate (fs=2fm) which can be used without overlapping is known as the **Nyquist rate**.

In most electronic DSP applications, performed by an A/D converter which also transforms the stream of samples into a binary code.



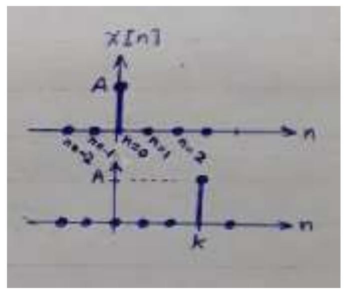
Fig(1.4)

1.3 Discrete -Time Signals (sequences):-

In digital signal processing , signals are represented as sequence of numbers called "samples" A sample value of a typical discrete- time signal or sequence is denoted as X [n] with the argument "n" being an integer in the range (-∞ and ∞). It should be noted that x[n] is defined only for integer values of "n" and undefined otherwhere. The most. common basic sequences are described next:

- **Unit Sample Sequence: -**

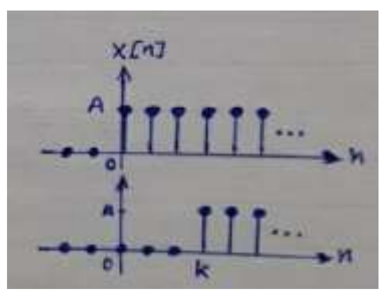
$$\delta[n] = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$\delta[n-k] = \begin{cases} A & n = k \\ 0 & n \neq k \end{cases}$$

- **Unit Step Sequence: -**

$$u[n] = \begin{cases} A & n \geq 0 \\ 0 & n < 0 \end{cases}$$

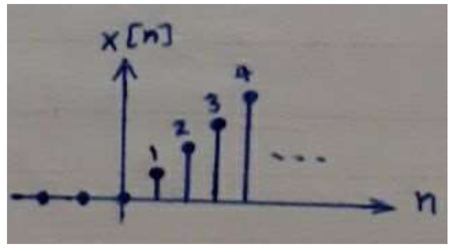


$$u[n-k] = \begin{cases} A & n \geq k \\ 0 & n < k \end{cases}$$

Note that $u[n] = \sum_{k=-\infty}^{n} \delta[k]$ and $\delta[n] = u[n] - u[n-1]$

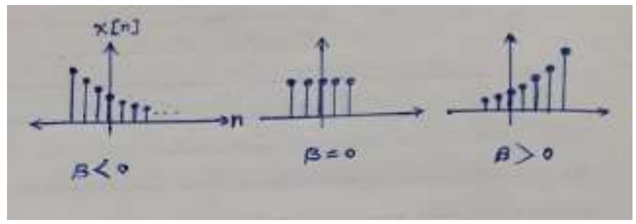
- **Unit Ramp Sequence: -**

$$r[n] = n\,u[n]$$



- **Exponential Sequence: -**

$$X[n] = Ae^{\beta n}$$



- **Sinusoidal Sequence: -**

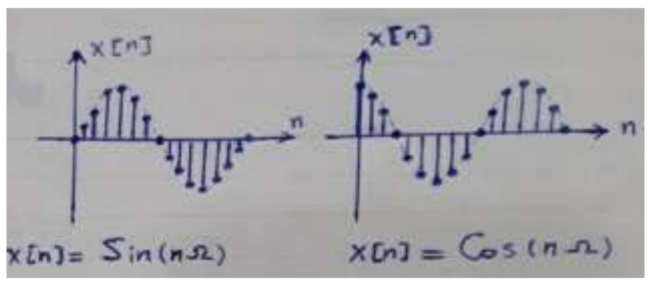
Analog sine $x(t) = \sin wt$

Discrete sine $x[n] = \sin(wnTs)$

Let wTs=Ω

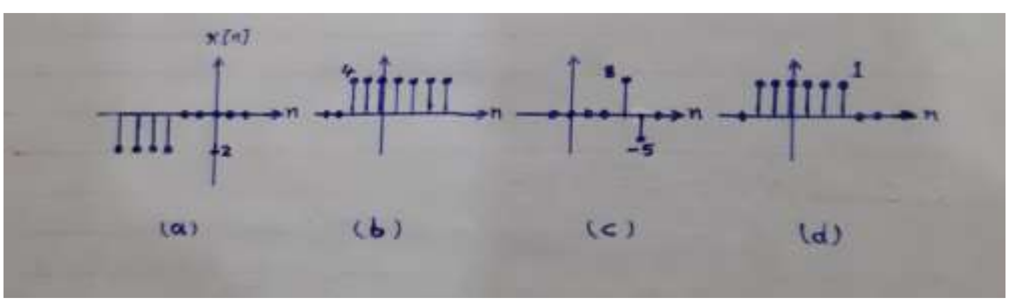
Where w=2 π fm

Ts=1/fs



Note that $xd[n] = x_a(t)|_{t=nT_s}$, usually we take Ts=1sec.

Example 1: find expressions for the various signals shown below



Solution:

a) x[n]= -2 u[-n-3]

b) x[n]= 4 u[n+2]

c) x[n]=8 δ[n-3]-5 δ[n-4]

d) x[n]= u[n+2]- u[n-4]

Example 2: sketch the following signals:

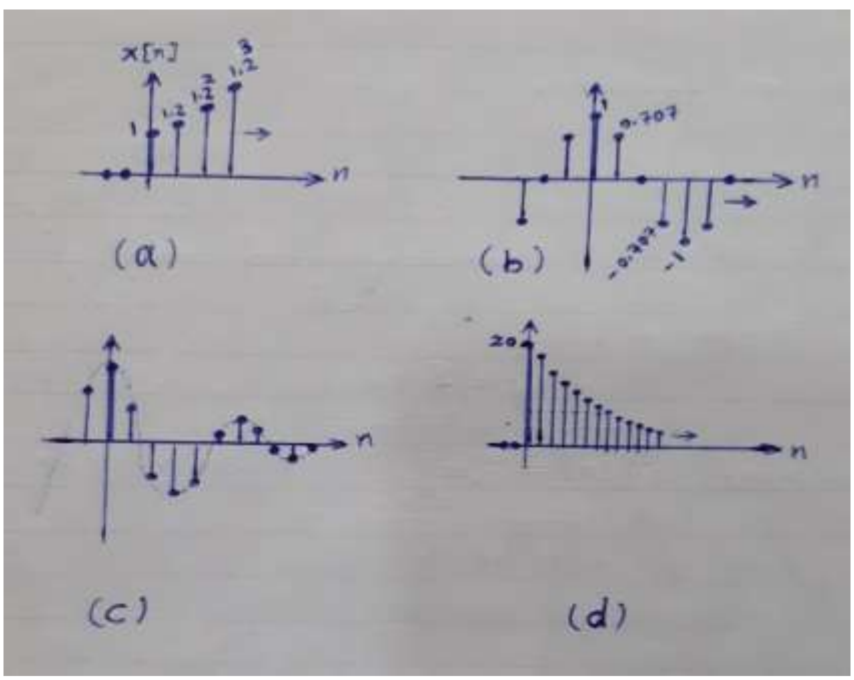
a) x[n]= $e^{0.2n}$ u[n]

b) x[n]= cos $(\frac{\pi n}{4})$

c) x[n]= $e^{-n/5}$ cos[n]

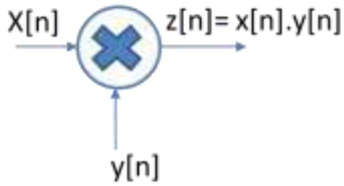
d) x[n]= 20 $(0.9)^n$ u[n]

solution:

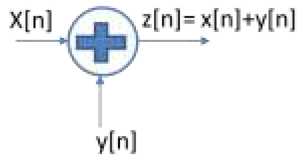


1.3.1 Operations on Sequences:

1. Modulation

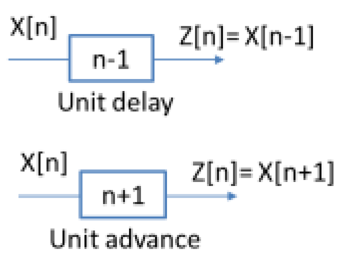


2. Addition



3. Scaling

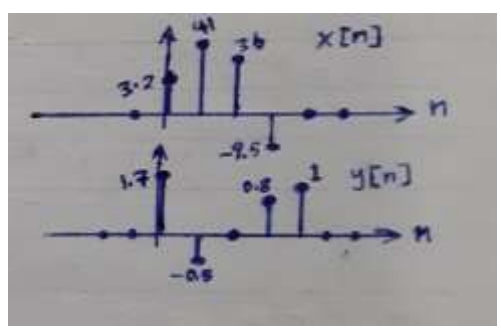
4. Time Shifting



Example: consider the following sequences of length (5) defined for (0≤n≤4).

x[n] ={ 3.2   41   36   -9.5   0 }

y[n] ={ 1.7 -0.5   0   0.8   1 }



Find a) x[n] . y[n]

b) x[n] + y[n]

c) $\frac{7}{2}$ x[n]

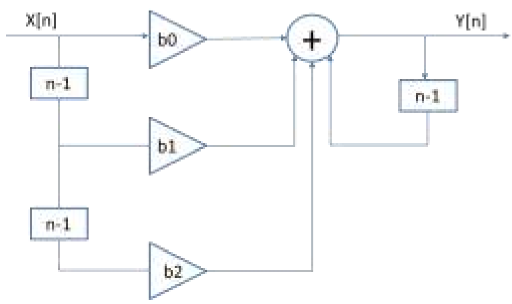
solution:

x[n] . y[n]={ 5.44   -20.5   0   -7.6   0 }

x[n]+y[n]={ 4.9   40.5   36   -8.7   1}

$\frac{7}{2}$ x[n]={ 11.2   143.5   126 -33.25 0}

Example: Analyze the discrete-time system shown below to determine the sequence y[n].



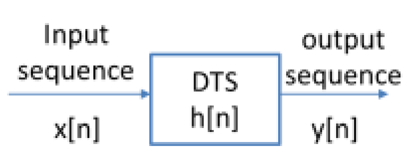
Solution:

$y[n]= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + y[n-1]$

This formula is known as ((***difference equation***))

1.4 Discrete -Time Systems (Digital Processors):-

The function of a discrete - time system is to process a given input sequence to generate an output sequence.



1.4.1 Classification of Discrete Time Systems:

The classification of DTS is based on the input - output relation of the system.

1- Linear System: It is the system for which the superposition principle always holds.

If $y_1[n]$ and $y_2[n]$ are the responses to the inputs $x_1[n]$ and $x_2[n]$ respectively , then for the input **$x[n] = \alpha x_1[n] + Bx_2[n]$** gives the output **$y[n] = \alpha y_1[n] + By_2[n]$**

Example: Test the linearity of the system :

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

Sol : By applying superposition theory:

$$= \frac{1}{3}(\alpha x_1[n+1] + B x_2[n+1] + x[n-1] + \alpha x_1[n] + B X_2[n] + \alpha x_1[n-1] + B x_2[n-1])$$

$$= \frac{1}{3}\alpha (x_1[n+1] + x_1[n] + x_1[n-1]) + \frac{1}{3}B (x_2[n+1] + x_2[n] + x_2[n-1])$$

$$= \alpha y_1[n] + B y_2[n]$$

⟹ The system is linear.

Example: The square -Law device $y[n]= x^2[n]$

$$(\alpha x_1[n] + B x_2[n])^2 = \alpha^2 x_1^2[n] + \underline{2 \alpha B x_1[n] x_2[n]} + \alpha^2 x_2^2[n]$$

$$\neq \alpha y_1[n] + B y_2[n]$$

⟹ The system is not linear.

2- Shift -Invariant System: (Time - invariant system)

If $y[n]$ is the response to an input $x[n]$ then the response to $x[n-n_0]$ is $y[n-n_0]$

3- Linear Time-Invariant System: (LTI)

It is the system that satisfies both the linearity and the time - invariance properties. Such systems are mathematically easy to analyze ,and easy to design.

4-Static and Dynamic System:

The system is said to be static or (memoryless) if its output depends up on the present input only.

5- Causal System:

In causal system, the output signal depends only on present and /or previous values of the input. The practical signal processors are always causal, because they cannot anticipate the future .

6- Invertible System:

If a digital system with input xln] gives an output y[n], then its inverse would produce x [n] if fed with y [n]. Most practical systems are invertible.

The LTI systems are also causal and invertible.

Example: Determine the following properties :(linearity, time invariance, causality, invertibility) for the systems :

(a) y [n] = 3x [n] - 4 x [n-1]

(b) y [n] = 2y [n-1] + x [n+2]

(c) y [n] =n x[n]

(d) y [n] = Cos (x[n])

Solution: (a) linear, time invariance, causal, invertible .

(b) linear, time invariance , not causal, invertible.

(c) linear , time - variance, causal, invertible.

(d) Non-linear, time invariance , causal, not invertible.

DSP

Chapter two

-Time Domain Analysis-

2.1 Introduction:

In this chapter we develop the basic techniques for describing digital signals and in the time domain. Such techniques are: impulse response, step response, and digital convolution.

2.2 The Impulse Response:

The response of digital system to sequence (x [n]=δ[n]) is called the unit sample response or simply "the impulse response", and is denoted as (h[n]).

Example 1: Find the impulse response of the system :

$y[n]=\frac{1}{3}x[n+1]+\frac{1}{3}x[n]+\frac{1}{3}x[n-1]$

Sol : we set x [n] = δ[n]

$y[n]=h[n]=\frac{1}{3}\delta[n+1]+\frac{1}{3}\delta[n]+\frac{1}{3}\delta[n-1]$

for n=-2 $\quad y[n]=\frac{1}{3}\delta[-1]+\frac{1}{3}\delta[-2]+\frac{1}{3}\delta[-3]=0$

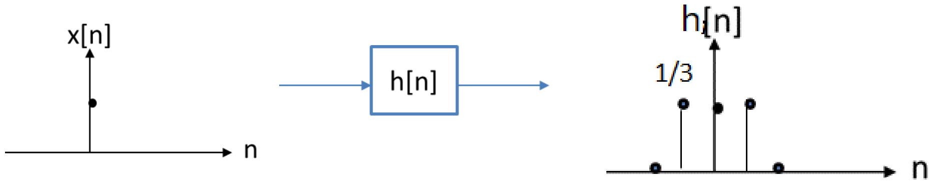
for n=-1 $y[n]=\frac{1}{3}\delta[0]+\frac{1}{3}\delta[-1]+\frac{1}{3}\delta[-2]=1/3$

for n=0 $y[n]=\frac{1}{3}\delta[1]+\frac{1}{3}\delta[0]+\frac{1}{3}\delta[-1]=1/3$

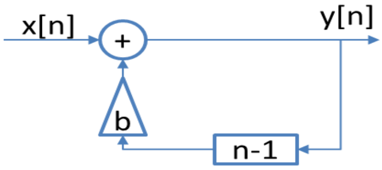
for n=1 $y[n]=\frac{1}{3}\delta[2]+\frac{1}{3}\delta[1]+\frac{1}{3}\delta[0]=1/3$

for n=2 $y[n]=\frac{1}{3}\delta[3]+\frac{1}{3}\delta[2]+\frac{1}{3}\delta[1]=0$

for $n\le -2$ and $n\ge 2$ ➡ $y[n]=0$



Example 2: Find the impulse response for the system shown below. Given **b=-0.9**



Sol:

$y[n]=-0.9\,y[n-1]+x[n]$

the impulse response $=h[n]=-0.9\,h[n-1]+\delta[n]$

$h[-1]=-0.9\,h[-2]+\delta[-1]=0+0=0$

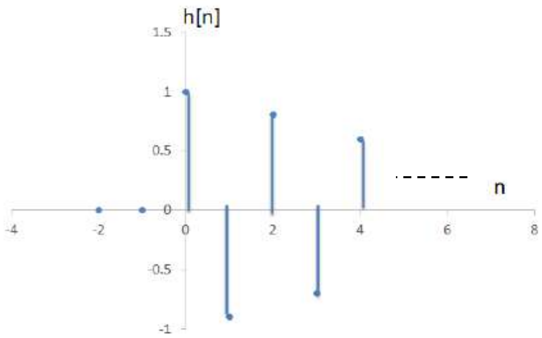
$h[0]=-0.9\,h[-1]+\delta[0]=0+1=1=(-0.9)^0$

$h[1]=-0.9\,h[0]+\delta[1]=-0.9 \quad =(-0.9)^1$

$h[2]=-0.9\,h[1]+\delta[2]=0.81 \quad =(-0.9)^2$

$h[3]=-0.9\,h[2]+\delta[3]=-0.729 \quad =(-0.9)^3$

$h[4]=-0.9\,h[3]+\delta[4]=0.656 \quad =(-0.9)^4$



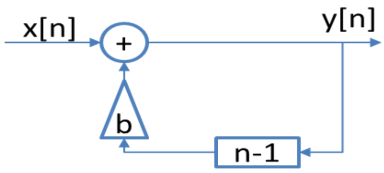
we can also find that $h[n]=(-0.9)^n\,u[n]$ or in general: $\mathbf{h[n]=b^n\,u[n]}$

2.3 The Step Response:

The response of a discrete-time system to a unit step sequence (x[n]=u[n]) is called the unit step response or simply the "step response", and is denoted as **S[n]**.

Example:

a) Find and sketch the step response for the system shown below. Given **b=0.8**.



b) Find the response to the rectangular pulse input bandlimited by ($0\leq n\leq 3$).

Sol: a) y[n]=0.8 y[n-1] + x[n]

For n<0 y[n]=0

For n=0 y[0]=0.8 y[-1] + x[0]=0+1=1

For n=1 y[1]=0.8 y[0] + x[1]=0.8(1)+1=1.8

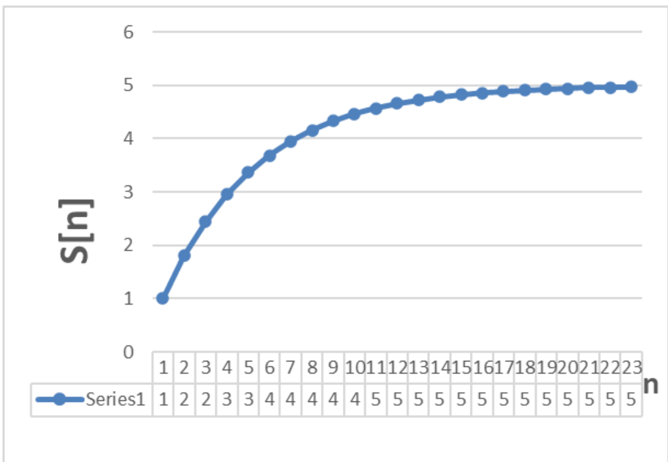
For n=2 y[2]=0.8 y[1] + x[2]=0.8(1.8)+1=2.44

For n=3 y[3]=0.8 y[2] + x[3]=0.8(2.44)+1=2.952

For n=∞ $y[\infty]=1+0.8^1+0.8^2+0.8^3+.......+0.8^\infty$

$=0.8^0+0.8^1+0.8^2+0.8^3+.......+0.8^\infty$

$=\sum_{n=0}^{\infty}(0.8)^n = \frac{1}{1-0.8}=5$ =steady state value



b) y[n]=0.8 y[n-1] +x[n]

for n<0 y[n]=0

n=0 y[n]=1

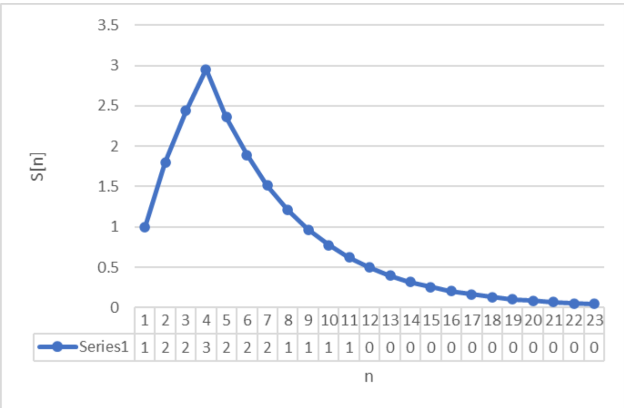
n=1 y[n]=1.8

n=2 y[n]=2.44

n=3 y[n]=2.952

n=4 y[n]=2.362

n=5 y[n]=1.89



- Note that increasing the value of **b** will increase the duration of the transient (the rise time).
- Transient response: it is the part of a response that vanishes as sample number approaches infinity.
- Steady state response: it is the part of the response that does not vanish as sample number approaches infinity.

2.4 Stability & Causality Conditions in Terms of the Impulse Response:

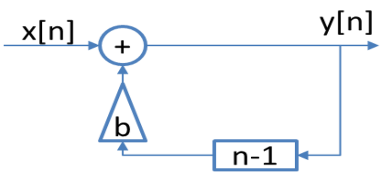
1. A digital system is stable **if and only if** ,the stability factor "**S**" is finite, i.e. **(S <∞).**where h[n]=impulse response of the system

$$S = \sum_{n=-\infty}^{\infty} |h[n]|$$

2. A digital system is said to be a causal **if and only if,** h[n]=0 for n<0.

   All physical systems are causal in that they do not react until a stimulus is applied.

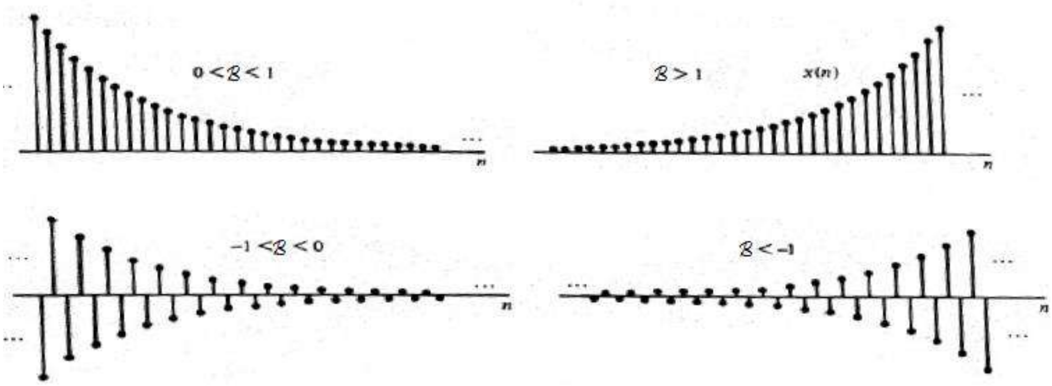
Example: check the stability and causality of the system shown below:



Sol:

$$S = \sum_{n=-\infty}^{\infty} |B^n u[n]| = \sum_{n=0}^{\infty} |B|^n = \frac{1}{1-|B|} \quad \text{for } |B|<1$$

The system is stable for |B|<1 or -1<B<1

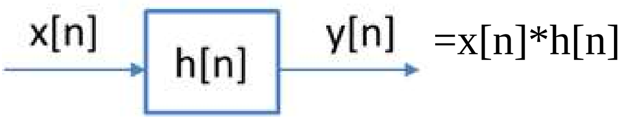


Since h[n]=0 for n<0 the system is causal.

DSP
Chapter two
-Time Domain Analysis-

2.5 Discrete Time Convolution

We have seen how to characterize LTI processors by their impulse or step responses. In practical cases, we need a general computer-based method to estimate a system's response to any form of input signal. The method which will do this is known as **"digital convolution"**



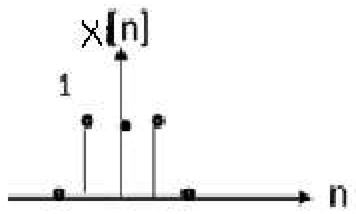
$$\boldsymbol{y}[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-\infty}^{\infty} \boldsymbol{x}[\boldsymbol{k}]\ \boldsymbol{h}[\boldsymbol{n}-\boldsymbol{k}]$$

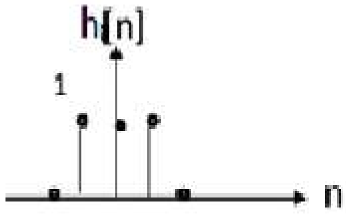
Which can be alternately as $\boldsymbol{y}[\boldsymbol{n}] = \sum_{\boldsymbol{k}=-\infty}^{\infty} \boldsymbol{x}[\boldsymbol{n}-\boldsymbol{k}]\ \ \boldsymbol{h}[\boldsymbol{k}]$

Example 1: convolution of two finite-duration sequence:

$x[n] = \begin{cases} 1 & for \quad -1 \le n \le 1 \\ 0 & otherwise \end{cases}$ $\qquad h[n] = \begin{cases} 1 & for \quad -1 \le n \le 1 \\ 0 & otherwise \end{cases}$

Sol:





y[n]=$x[n] * h[n]$ =$\sum_{k=-\infty}^{\infty} \boldsymbol{x[k]\, h[n-k]}$

for n=-3 y[-3]=$x[-1]\, h[-2] + x[0]\, h[-3] + x[1]\, h[-4]$=0

for n=-2 y[-2]=$x[-1]\, h[-1] + x[0]\, h[-2] + x[1]\, h[-3]$=1

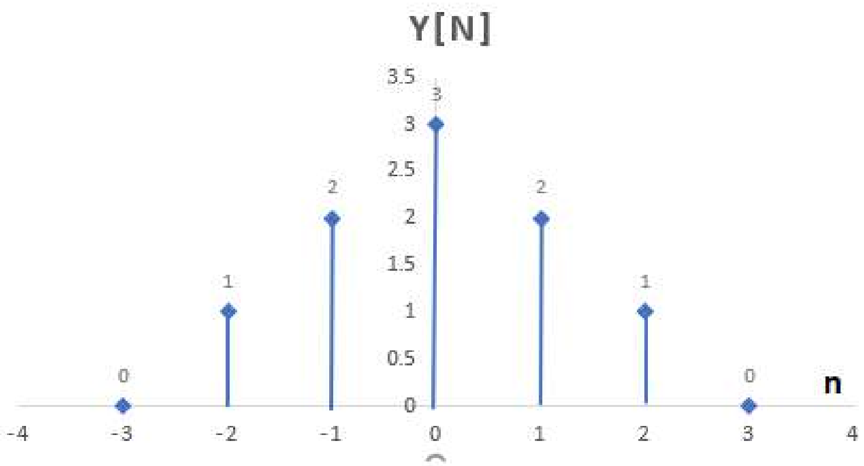
for n=-1 y[-1]=$x[-1]\, h[0] + x[0]\, h[-1] + x[1]\, h[-2]$=2

for n=0 y[0]=$x[-1]\, h[1] + x[0]\, h[0] + x[1]\, h[-1]$=3

for n=1 y[1]=$x[-1]\, h[2] + x[0]\, h[1] + x[1]\, h[0]$= 2

for n=2 y[2]=$x[-1]\, h[3] + x[0]\, h[2] + x[1]\, h[1]$= 1

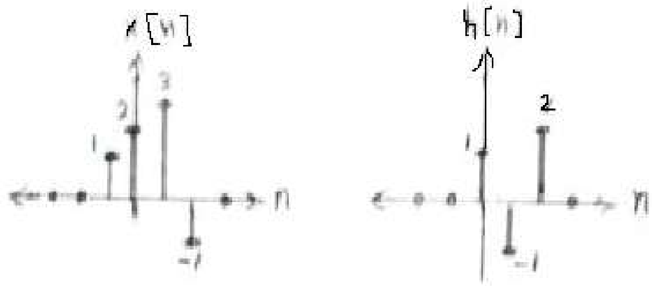
for n=3 y[3]=$x[-1]\, h[4] + x[0]\, h[3] + x[1]\, h[2]$= 0



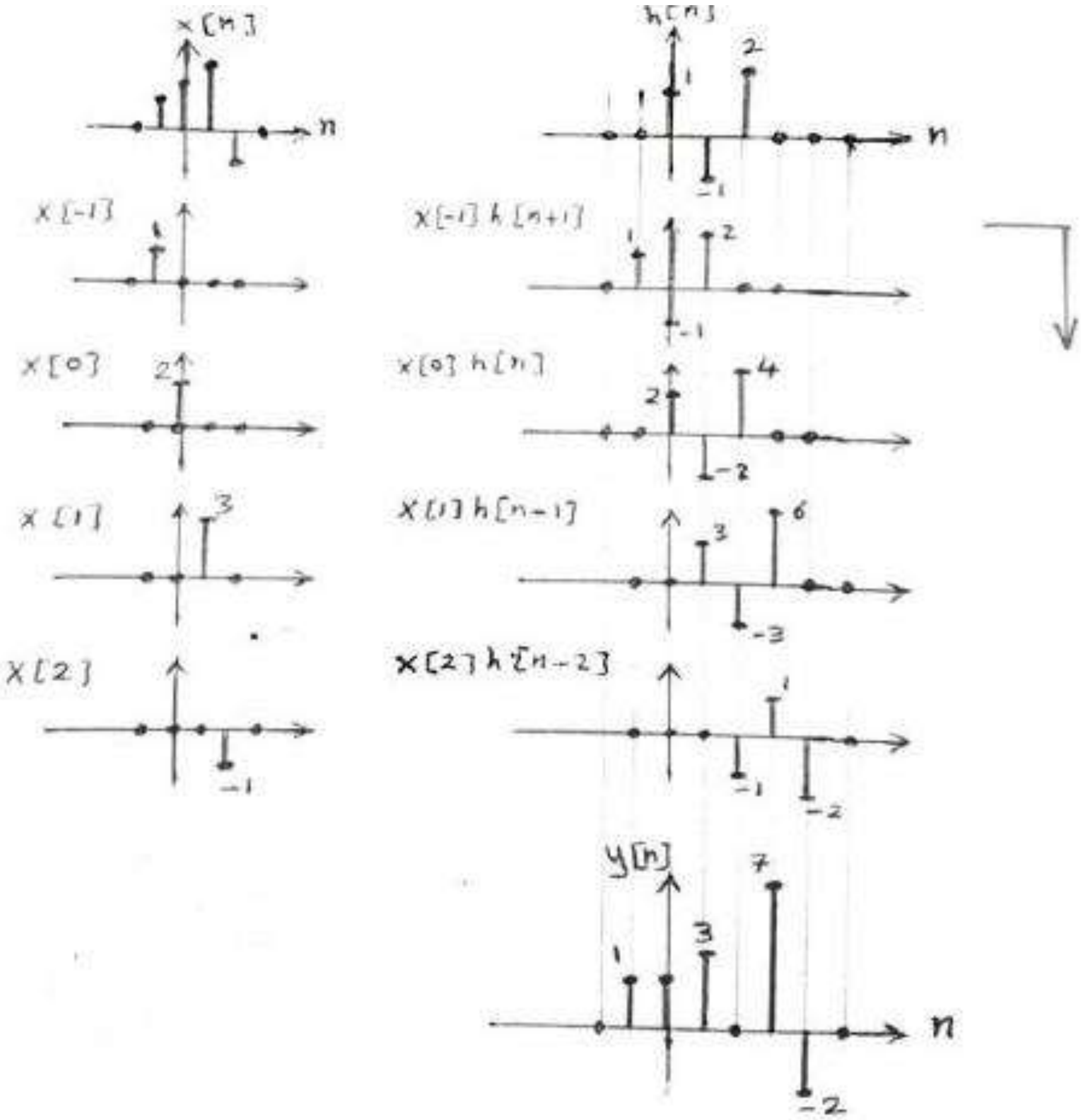
Example 2: Find x[n] * h[n] where:

x[n]=[1 2 3 -1]
↑

h[n]=[1 -1 2]



Sol 1: we can solve the problem **numerically** as in the previous example or by using the **graphical method** as shown below:



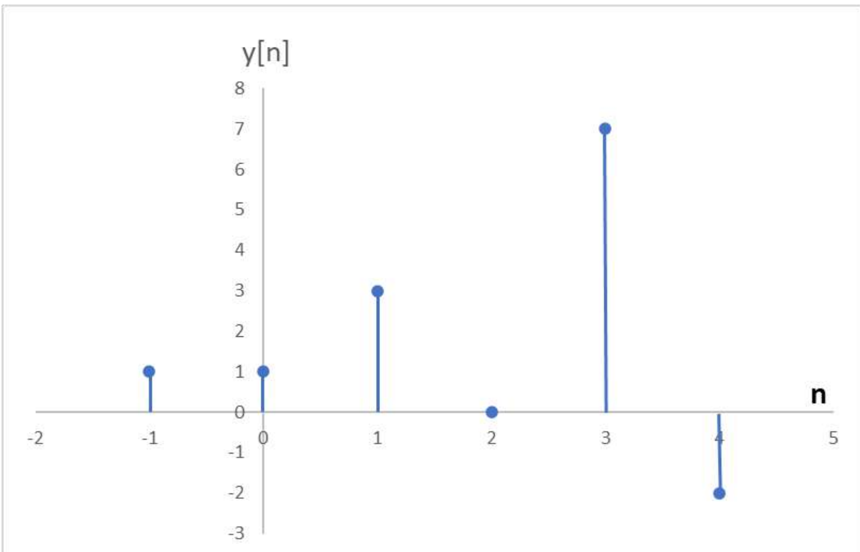
Sol 2: Also this problem can be solved using **the multiplication method** as shown below:

```
x[n]          1   2   3  -1
h[n]      ×   1  -1   2
-------------------------------
              2   4   6  -2
          -1  -2  -3   1
+      1   2   3  -1
-------------------------------
y[n]=  1   1   3   0   7  -2
           ↑
```

**and it can also be solved by using the 4th method tabulation method

	h[0]	h[1]	h[2]
x[-1]	1	-1	2
x[0]	2	-2	4
x[1]	3	-3	6
x[2]	-1	1	-2

Y[n]	Y[-1]	Y[0]	Y[1]	Y[2]	Y[3]	Y[4]
=	1	1	3	0	7	-2



Example: convolution of an infinite -duration sequence with a finite -duration sequence . Given

$x[n] = \begin{cases}(n+1) & for\ 0 \le n \le 2 \\ 0 & otherwise\end{cases}$ $\qquad h[n] = a^n u[n]$

Sol:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \; h[n-k]$$

$$y[n] = \sum_{k=0}^{2} (k+1) \; a^{n-k} \, u[n-k]$$

$$y[n] = a^n \, u[n] + 2 \, a^{n-1} \, u[n-1] + 3 \, a^{n-2} \, u[n-2]$$

Example: convolution of two infinite -duration sequences

$$x[n] = a^n u[n] \qquad\qquad h[n] = b^n u[n]$$

Sol: $y[n] = \sum_{k=-\infty}^{\infty} a^k u[k] \; b^{n-k} \, u[n-k]$

$$=\sum_{k=0}^{n} a^k \; b^{n-k} = \sum_{k=0}^{n} a^k \; b^n b^{-k} = b^n \sum_{k=0}^{n} (\tfrac{a}{b})^k$$

$$\therefore y[n] = b^n \, \frac{1-(\frac{a}{b})^{n+1}}{1-(\frac{a}{b})}$$

2.6 Digital Convolution Properties:

The convolution operation satisfies several useful properties:

1. Commutative: $x_1[n] \otimes x_2[n] = x_2[n] \otimes x_1[n]$
2. Associative: $(x_1[n] \otimes x_2[n]) \otimes x_3[n] = x_1[n] \otimes (x_2[n] \otimes x_3[n])$
3. Distributive: $x_1[n] \otimes (x_2[n] + x_3[n]) = x_1[n] \otimes x_2[n] + x_1[n] \otimes x_3[n]$

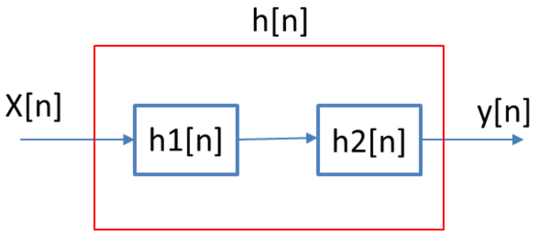
2.7 Simple Interconnection Schemes:

Two widely used schemes for developing complex LTI systems from simple LTI section:

1. Cascade Connection

In figure below the two sections are said to be connected in cascade. The overall impulse response h[n] is given by

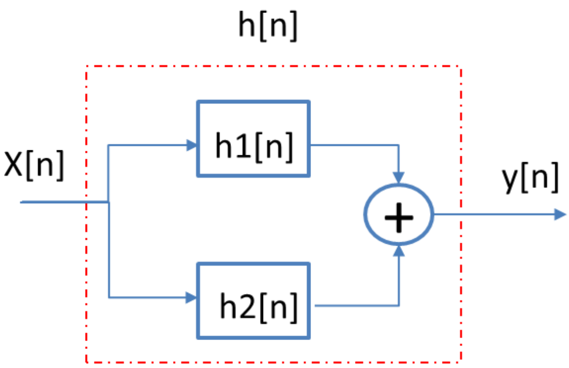
$$h[n]=h_1[n] \otimes h_2[n]$$



2. Parallel Connection

The connection scheme of figure below is called parallel connection. The impulse response of the overall system is given here by:

$$h[n]=h_1[n]+h_2[n]$$



2.8 Classification of LTI Systems:

LTI systems are usually classified according to the length of their impulse response. If h[n] is of finite length ,(i.e. h[n]=0 for $n<N_1$ and $n>N_2$ where $N_2>N_1$), then the system is known as a "finite impulse response" or FIR system.

While if h[n] of a system is of infinite length then the system is known as an "infinite impulse response" or IIR system.

DSP
Chapter three
-Frequency Domain Analysis-

3.1 Introduction

In previous chapter we pointed out that any arbitrary sequence can be represented in the time-domain as a weighted linear combination of delayed unit sample sequences (δ[n-k]). we consider in this chapter an alternate description of a sequence in terms of complex exponential sequences of the form (e^{-jwn}) and (z^{-n}). This leads to two particularly useful representations of discrete sequences and systems in a transform domain: -

1- Discrete Fourier transform (DFT).

2- Z-transform.

3.2 The Discrete Fourier Transform: -

In many signal processing applications, the distinguishing features of signals and systems are most easily interpreted in the frequency domain. The main analytic tool used to transform from time domain (x [n]) to frequency domain (X[k]) is the **Fourier transform**. The DFT is important for two reasons : First, it allows us to determine the frequency content of a signal , that is , to perform spectral analysis . The second is to perform filtering operations in the frequency domain .

Periodic sequences can be represented in the frequency domain by means of a " **discrete Fourier Series**", which is given by :

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j2\pi kn/N}$$

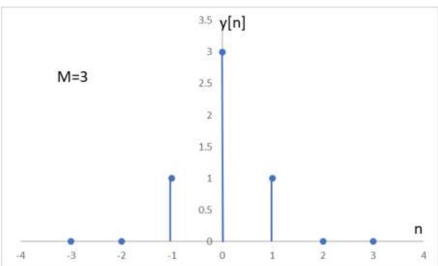
where x[n] is a periodic sequence with period (N), and the spectral coefficients X˜[k] are evaluated for (0≤k≤N-1) , i.e X˜[k] is also periodic with a period N). The inverse of above process which allows us to recover the signal from it's spectrum, is given by:

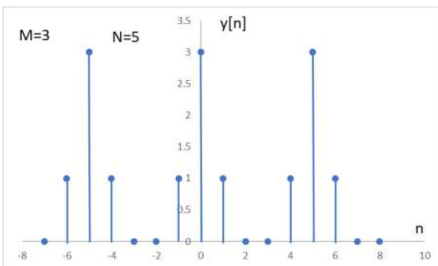
$$\tilde{x}[n] = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{X}[k]\, e^{j2\pi kn/N}$$

Truly periodic signals are rarely encountered in practical DSP. Non-periodic (aperiodic) signals with a finite number of nonzero sample values are the more Common.

Let x [n] be aperiodic signal with duration containing (M) samples. We will consider it is periodic signal of period (N) where N≥ M then: -

$$\tilde{x}[n] = \begin{cases} x[n] & for\ \ 0 \le n \le M-1 \\ 0 & for\ \ M \le n \le N-1 \end{cases}$$





"The sequence X [n] is called the " periodic extension "of x[n]. We are free to choose the value of (N) , but we must be careful not to be too small, because ; if N<M, an overlap then occurs when periodic extension is formed .

Summarizing all the above, we arrive at the desired

DFT relationship :

for $0 \le k \le N-1$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}$$

$$= \underbrace{\sum_{n=0}^{N-1} x[n] \cos \frac{2\pi kn}{N}}_{\text{Re } X[k]} - \underbrace{j \sum_{n=0}^{N-1} x[n] \sin \frac{2\pi kn}{N}}_{\text{Im } X[k]}$$

The inverse discrete fourier transform or IDFTis given by:

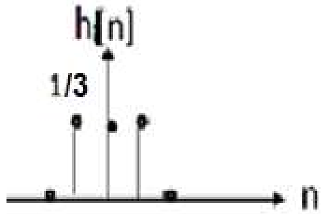
$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k]\, e^{j2\pi kn/N}$$

Sometimes we assume $e^{-j2\pi/N} = W_N$, thus $e^{-j2\pi kn/N} = W_N^{nk}$

The two equation form the basis of the computer algorithms that evaluate the DFT.

Ex: Compute the N-point DFT, were N=3 for the sequence :

$$\boldsymbol{h[n]} = \begin{cases} \frac{1}{3} & \boldsymbol{for}\ \ 0 \le n \le 2 \\ 0 & \boldsymbol{otherwise} \end{cases}$$



Sol: $$H[k] = \sum_{n=0}^{3-1} h[n]\, e^{-\frac{j2\pi kn}{N}}$$

$$H[k] = \frac{1}{3} + \frac{1}{3}e^{-\frac{j2\pi k}{N}} + \frac{1}{3}e^{-\frac{j4\pi k}{N}}$$

$$H[0] = \frac{1}{3} + \frac{1}{3}(\,1\,) + \frac{1}{3}(\,1\,) = 1$$

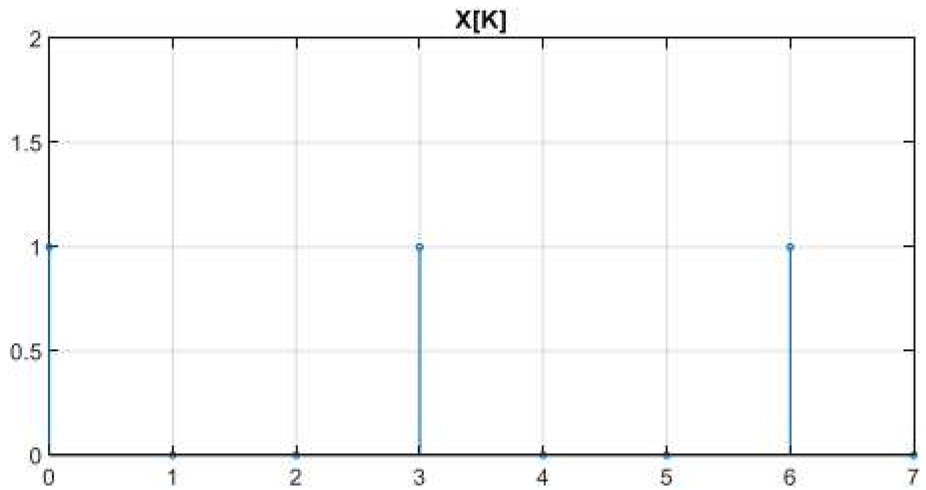
$$H[1] = \frac{1}{3} + \frac{1}{3}e^{-\frac{j2\pi}{3}} + \frac{1}{3}e^{-\frac{j4\pi}{3}} = 0$$

$$H[2] = \frac{1}{3} + \frac{1}{3}e^{-\frac{j4\pi}{3}} + \frac{1}{3}e^{-\frac{j8\pi}{3}} = 0$$

$$H[3] = 1$$

⋮

Hence X[k]=[1 0 0 1 0 0 1 0........]



Ex: Compute DFT for the infinite-duration sequence given by:

$$h[n]=a^n\, u[n]$$

Sol:

$$H[k] = \sum_{n=0}^{N-1} h[n]\, e^{-\frac{j2\pi kn}{N}}$$

$$H[k] = \sum_{n=0}^{\infty} a^n\, e^{-\frac{j2\pi kn}{N}}$$

$$H[k] = \sum_{n=0}^{\infty} (\mathrm{a}\, e^{-\frac{j2\pi k}{N}})^n$$

$$H[k] = \frac{1}{1-\mathrm{a}\, e^{-\frac{j2\pi k}{N}}}$$

The following table shows some useful DFT pairs:

X[n]	X[k]
δ[n]	1
δ[n-a]	$e^{\frac{-j2\pi k}{N}a}$
$a^n u[n]$	$\frac{1}{1-ae^{\frac{-j2\pi k}{N}}}$
u[n]	$\frac{1}{1-e^{\frac{-j2\pi k}{N}}}$
$e^{an} u[n]$	$\frac{1}{1-e^{a}e^{\frac{-j2\pi k}{N}}}$

Note: Sometimes the equation of X[k] includes real plus imaginary parts so we convert it into magnitude and phase then sketch the magnitude function.

3.2.1 Properties of DFT:

1- **Linearity:**

DFT (Ax_1[n] + Bx_2[n]) = AX_1[k] + BX_2[k]

2- Convolution:

DFT (x[n] $\otimes$ y[n]) = X[k] **.** Y[k]

3- Modulation:

DFT (x[n] **.** y[n]) = X[k] $\otimes$ Y[k]

4- Periodicity:

X[k] = X[k+N]

5- Time Shifting:

DFT (x[n-a]) = X[k] $e^{-\frac{j2\pi k\ a}{N}}$

6- Frequency Shifting:

DFT (x[n] $e^{-\frac{j2\pi k\ a}{N}}$) = X[(k-a)N] = $\begin{cases} x[k-a] & a \le k \le N-1 \\ x[k-a+N] & 0 \le k \le a \end{cases}$

7- Parseval's Theorem

The power in discrete time domain is the same one in the discrete frequency domain or

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2$$

Ex: prove that DFT of 1. $\delta[n]=1$

2. $\delta[n-n_0] = W_N^{kn0}$

Sol:

$$X[k] = \sum_{n=0}^{\infty} x[n]\, e^{-\frac{j2\pi kn}{N}}$$

1. Since $x[n]=\delta[n]=\begin{cases} 1 & n=0 \\ 0 & otherwise \end{cases}$

   $X[k]= 1 \cdot e^0+0+0 \quad =1$

2. $X[k] = \sum_{n=0}^{\infty} \delta[\mathrm{n}-\mathrm{n0}]\, e^{-\frac{j2\pi kn}{N}} = e^{-\frac{j2\pi kn0}{N}} = W_N^{kn0}$

Ex: Find the power in time and frequency domain for

$x[n]=\frac{1}{3}\delta(\mathrm{n}) + \frac{1}{3}\delta(\mathrm{n}-1) + \frac{1}{3}\delta(\mathrm{n}-2)$

Sol: The energy in the time domain

$=\sum_{n=0}^{N-1}|x[n]|^2 = (\frac{1}{3})^2+(\frac{1}{3})^2+(\frac{1}{3})^2=\frac{1}{3}$

We previously found that x[k]=[1 0 0]

The energy in the frequency domain $=\frac{1}{N}\sum_{k=0}^{N-1}|x[k]|^2$

$=\frac{1}{3}(1^2+0^2+0^2)= \frac{1}{3}$

3.3. The inverse DFT (IDFT):

To transfer the frequency response into the corresponding discrete time sequence, we use the following formula:

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]\, e^{j2\pi kn/N}$$

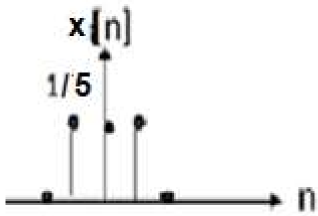
Ex: compute the IDFT for $H[k] = \frac{1+2\cos\frac{2\pi k}{N}}{5}$

Sol: we have $h[n] = \frac{1}{N}\sum_{n=0}^{N-1} H[k]\, e^{j2\pi kn/N}$

Or we can directly find the IDFT from the table

Since x[k]=$\frac{1}{5} + \frac{2}{5}\left(\frac{e^{\frac{j2\pi k}{N}} + e^{-\frac{j2\pi k}{N}}}{2}\right)$

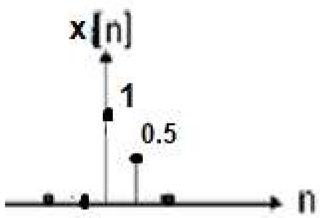
$=\frac{1}{5} + \frac{1}{5}e^{\frac{j2\pi k}{N}} + \frac{1}{5}e^{-\frac{j2\pi k}{N}}$

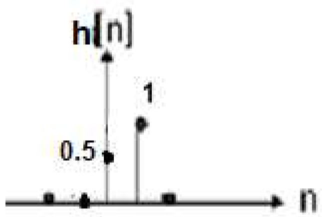


x[n]= $\frac{1}{5}\delta[n] + \frac{1}{5}\delta[n+1] + \frac{1}{5}\delta[n-1]$

Ex: Perform the linear convolution with DFT.

$$x[n] = \begin{cases} 1 & for\ n = 0 \\ 0.5 & for\ n = 1 \\ 0 & otherwise \end{cases} \qquad h[n] = \begin{cases} 0.5 & for\ n = 0 \\ 1 & for\ n = 1 \\ 0 & otherwise \end{cases}$$





Sol:

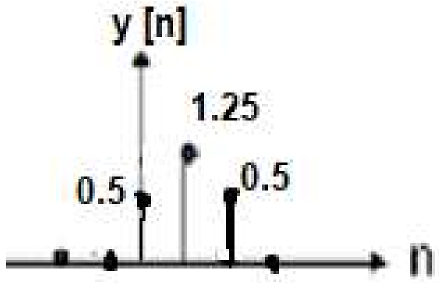
$$X[k] = \sum_{n=0}^{1} x[n]\, e^{-\frac{j2\pi kn}{N}}$$

= x[0]+x[1] $e^{-\frac{j2\pi k}{N}}$

X[k]=1+0.5$e^{-\frac{j2\pi k}{N}}$ Also H[k]=0.5+$e^{-\frac{j2\pi k}{N}}$

Y[k] = X[k] . H[k] = 0.5+1.25$e^{-\frac{j2\pi k}{N}} + 0.5e^{-\frac{j2\pi k(2)}{N}}$

$y[n] = IDFT\ (Y[k]) = 0.5\delta[n] + 1.25\ \delta[n-1] + 0.5\delta[n-2]$



DSP
Chapter three
-Frequency Domain Analysis-

3.3 The Z-Transform

The Z- transformation does for analysis and design of digital systems what the laplace transformation does for analysis and design of analog systems.

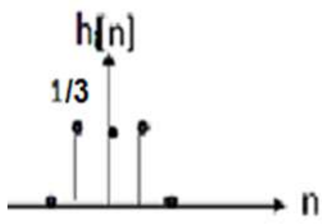
The Z- transform of a digital signal x[n] is defined as:

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

The Z- transform and the DFT are closely related

$$X(z) = X[k]\big|_{e^{j2\pi k/N} = z}$$

Ex: Find the Z- transform of the signal shown below:



Sol:

$$X(z) = \sum_{n=-1}^{1} x[n]\, z^{-n}$$

$$= \frac{1}{3} Z + \frac{1}{3} + \frac{1}{3} Z^{-1}$$

Ex: Find the Z- transform of exponentially signal given by:

$x[n] = a^n\, u[n]$

Sol: $X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} a^n\, z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$

By applying the geometric sum formula

$$X(z) = \frac{1}{1 - az^{-1}} \quad OR \quad X(z) = \frac{z}{z - a}$$

for $|az^{-1}| < 1$.

This is some geometric sum formula (just for know)

$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$	$\sum_{n=0}^{\infty} a^n = \frac{1}{1 - a} \quad \|a\| < 1$
$\sum_{n=0}^{N-1} na^n = \frac{(N-1)a^{N+1} - Na^N + a}{(1-a)^2}$	$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2} \quad \|a\| < 1$
$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$	$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N(N-1)(2N-1)$

Note: the Z-transform pairs of some important signals:

X[n]	X(Z)
δ[n]	1
δ[n-a]	$z^{-a} = \frac{1}{z^a}$
u[n]	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$
a^n u[n]	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$
r[n]	$\frac{z}{(z-1)^2}$
x[n-a]	$z^{-a} X(z)$

3.3.1 Inverse Z-Transform: -

The inverse transformation of a function X(z) is defined as:

$$x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1} dz$$

Where the circular symbol on the integral sign denotes a closed contour in the complex plane such integration is rather difficult and beyond our scope. Fortunately, several simpler approaches are available. Two simple methods for the inverse transform computation are reviewed in the next two examples.

Ex: A signal has the z-transform $X(z) = \frac{1}{z\,(z-1)(2z-1)}$ use the method of **partial fraction** to recover the signal x[n].

Sol:
$$X(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{2z-1}$$

$$A = \lim_{z\to 0} \frac{1}{(z-1)(2z-1)} = 1$$

$$B = \lim_{z\to 1} \frac{1}{z(2z-1)} = 1$$

$$C = \lim_{z\to 0.5} \frac{1}{z(z-1)} = -4$$

$$X(z) = \frac{1}{z} + \frac{1}{z-1} - \frac{4}{2z-1}$$

$$X(z) = \frac{1}{z} + \frac{1}{z-1} - \frac{2}{z-0.5}\} * (z\,z^{-1})$$

$$X(z) = z^{-1} + z^{-1}\frac{z}{z-1} - z^{-1}\,(2)\frac{z}{z-0.5}$$

Referring to the table ➡

$$x[n] = \delta[n-1] + u[n-1] + 2\;0.5^{n-1}\,u[n-1]$$

Ex: Solve the previous example using **long division** of the numerator by the denominator $X(z) = \frac{1}{z\,(z-1)(2z-1)}$

Sol: $X(z) = \frac{1}{z\,(z-1)(2z-1)} = \frac{1}{2z^3-3z^2+z}$

$$\begin{array}{r|l} & 0.5z^{-3} + 0.75z^{-4} + 0.875z^{-5} + \cdots \\ \hline 2z^3 - 3z^2 + z & 1 \end{array}$$

$$1 - 1.5z^{-1} + 0.5z^{-2}$$

$$\overline{1.5z^{-1} - 0.5z^{-2}}$$

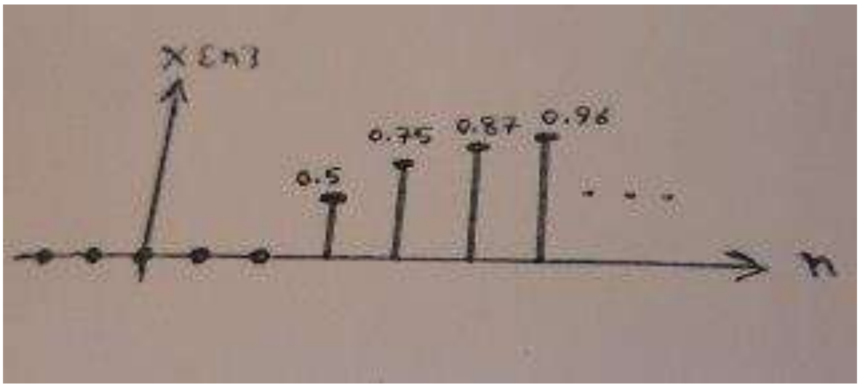
$$1.5z^{-1} - 2.25z^{-2} + 0.75z^{-3}$$

$$\overline{1.75z^{-2} + 0.75z^{-3}}$$

$$\vdots \qquad \vdots$$

$$X(z) = 0.5z^{-3} + 0.75z^{-4} + 0.875z^{-5} + \cdots$$

$$x[n] = 0.5\delta[n-3] + 0.75\delta[n-4] + 0.875\delta[n-5] + \cdots$$



3.3.2 Z-Transform Properties: -

1. **Linearity:**

$$A\,x_1[n] + B\,x_2[n] \;\leftrightarrow A\,X_1(z) + B\,X_2(z)$$

2. **Time – Shifting:**

$$x[n-a] \;\leftrightarrow X(z)\,z^{-a}$$

3. Convolution:

$$x_1[n] * \;\; x_2[n] \;\leftrightarrow\;\; X_1(z)\,.\,X_2(z)$$

4. Differentiation:

$$n\,x[n] \;\leftrightarrow (-z)\,\frac{d\,X(z)}{dz}$$

5. Multiplication by an exponential sequence:

$$a^n\,x[n] \leftrightarrow X(\frac{z}{a})$$

Ex: Determine the z-transform of the sequence given by:

$$y[n] = (n+1)\,\alpha^n u[n]$$

Sol:

$$y[n] = \alpha^n u[n] + n\,\alpha^n u[n]$$

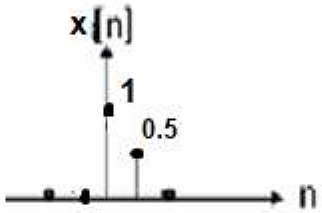
$$\alpha^n u[n] \leftrightarrow \frac{1}{1-\alpha z^{-1}} \quad , \; |\alpha z^{-1}| < 1$$

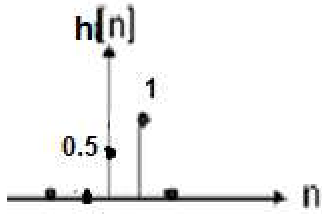
$$\therefore Y(z) = \frac{1}{1-\alpha z^{-1}} + (-z)\frac{d}{dz}\frac{1}{1-\alpha z^{-1}}$$

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{\left(1-\alpha z^{-1}\right)^2}$$

$$Y(z) = \frac{1}{\left(1-\alpha z^{-1}\right)^2} \qquad \text{for } |\alpha z^{-1}| < 1$$

Ex: Perform the linear convolution with z-transform.





Sol: $x[n] * h[n] \leftrightarrow X(z)\,.H(z)$

$$X(z) = \sum_{n=0}^{1} x[n]z^{-n} = 1 + 0.5z^{-1}$$

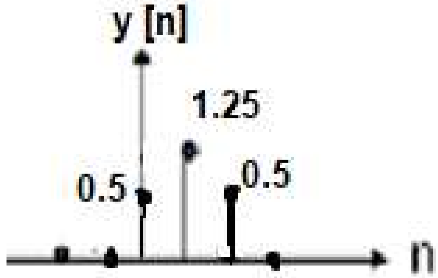
$$H(z) = \sum_{n=0}^{1} h[n]z^{-n} = 0.5 + z^{-1}$$

$$Y(z) = (1 + 0.5z^{-1})\,(0.5 + z^{-1})$$

$$Y(z) = 0.5 + 1.25\, z^{-1} + 0.5z^{-2}$$

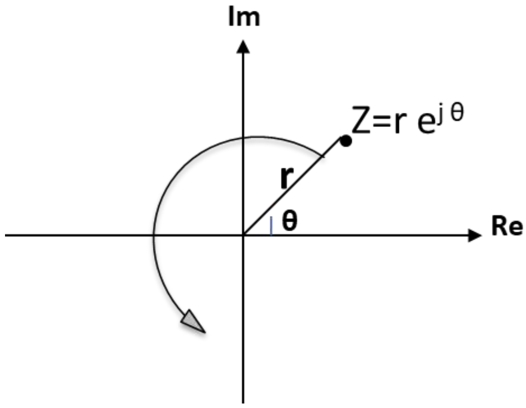
$$y[n] = Inverse\ (Y[z])$$

$$y[n] = 0.5\delta[n] + 1.25\,\delta[n-1] + 0.5\delta[n-2]$$



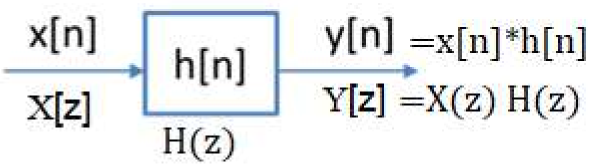
3.3.3 Z- Plane: -

The independent variable (z) is a complex variable. Values of (z) can be associated with points in a plane called the z-plane. The z-plane is an important graphical tool in the theory and application of the z-transformation.



3.3.4 Evaluation of LTI system Response using Z-Transform: -

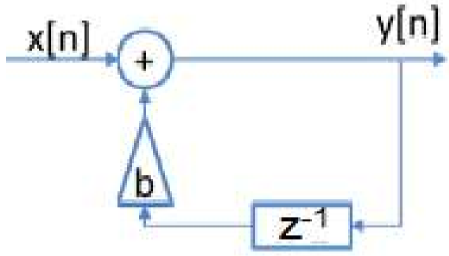
The output of LTI for input x[n] can be obtained using z-transformation: $y[n]=\mathcal{Z}^{-1}\,Y(z)=\mathcal{Z}^{-1}\,X(z)\,H(z)$



Where h[n]=impulse response of the system=$y[n]|_{x[n]=\delta[n]}$

H(z)=transfer function of the system $=\mathcal{Z}$ h[n] or $=\frac{Y(z)}{X(z)}$

Ex: Find the impulse response and the transfer function of the following system.



Sol:

$$y[n] = b\,y[n-1] + x[n]$$

$$Y(z) = b\,z^{-1}Y(z) + X(z)$$

For the impulse response x[n]=δ[n] ➡ $X(z) = 1$

$$\therefore Y(z) = b\,z^{-1}Y(z) + 1$$

$$H(z) = \frac{1}{1 - bz^{-1}}$$

$$h[n] = y[n]|_{x[n]=\delta[n]} = \mathcal{Z}^{-1} H(z) = b^n\,u[n]$$

or directly $T.F. = H(z) = \frac{Y(z)}{X(z)} = \mathcal{Z}$ h[n] $= \frac{1}{1-bz^{-1}}$

3.3.5 Stability Determination Based Z-Transform: -

A digital signal or an LTI system can always be described using z-transform as the ratio: -

$$X(z) = \frac{N(z)}{D(z)} = \frac{k\,(z - z_1)(z - z_2)\ldots}{(z - p_1)(z - p_2)\ldots}$$

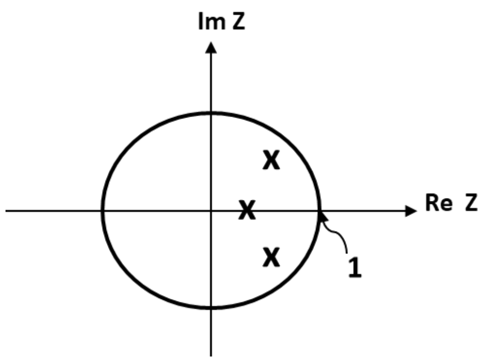
The constant z_1, z_2, z_3... are called the **"zeros"** of X(z), because they are the values of (z) for which X(z)is zero. Conversely p_1, p_2, p_3 ... are known as the **"poles"** of X(z). the poles and zeros are either real or occur in complex conjugate pairs. ***"The digital system is stable, if and only if, all the poles of the system lie inside the unit circle in the z-plane"*** k=the system gain

Ex: check the stability of the system given by:

$$\boldsymbol{H(z) = \frac{k\,(z-1)^2}{(z-0.3)(z^2 - z + 0.5)}}$$

Sol:

$$\boldsymbol{H(z) = \frac{k\,(z-1)^2}{(z-0.3)(z-0.5+j0.5)(z-0.5-j0.5)}}$$



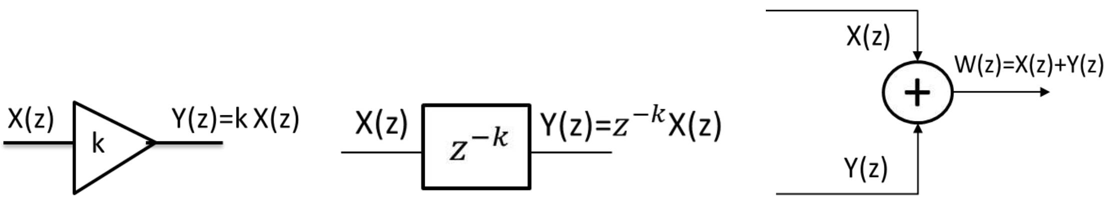
Since all the poles lie inside the unit circle ➡ The system is stable.

3.3.6 Digital System Implementation from its Function: -

Since the z-transform is a linear transformation, the system implementation procedure is similar to that in the time domain. The most convenient form for system synthesis is the z-transform of the general difference equation given by:

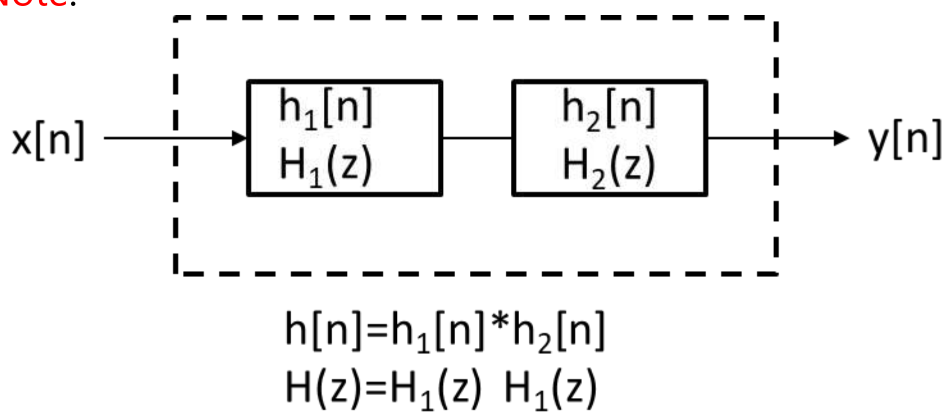
$$Y(z) = \sum_{k=1}^{M} a_k \, z^{-k} \, Y(z) + \sum_{k=a}^{b} a_k \, z^{-k} \, X(z)$$

This equation can be implemented using the following symbols for elementary LTI system



Gain (amplification) Delay Addition

Note:



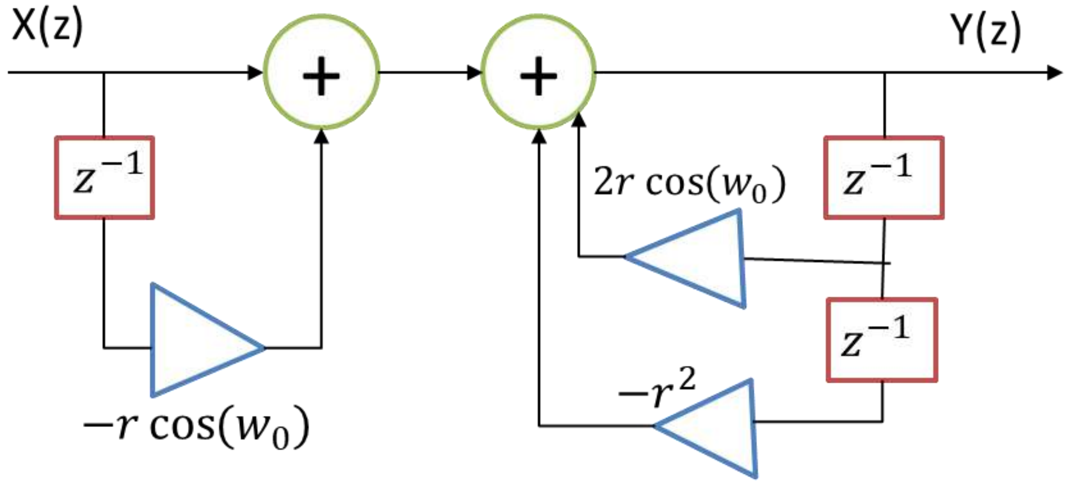
Ex: Implement the 2nd order recursive filter :

$$y[n] = 2\,r\cos(w_0)\,y[n-1] - r^2 y[n-2] + x[n] - r\cos(w_0)\,x[n-1]$$

Sol:

$$Y(z) = 2\,r\cos(w_0)\,z^{-1}Y(z) - r^2 z^{-2}Y(z) + X(z) - r\cos(w_0)\,z^{-1}X(z)$$

The structure of the filter is shown below



Ex: Realize the digital system given by :

$$H(z) = \frac{z(z+1)}{(z^2 - z + 0.5)}$$

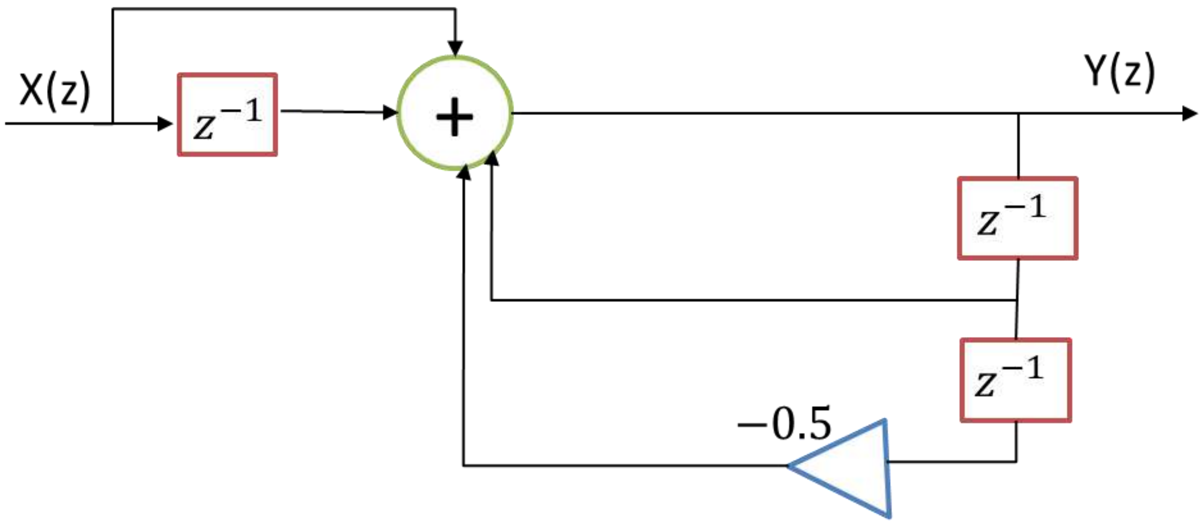
Sol:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{(z^2 - z + 0.5)}\;] * \frac{z^{-2}}{z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

$$Y(z)(1 - z^{-1} + 0.5z^{-2}) = X(z)(\,1 + z^{-1})$$

$$Y(z) = X(z) + z^{-1}X(z) + z^{-1}Y(z) - 0.5z^{-2}Y(z)$$

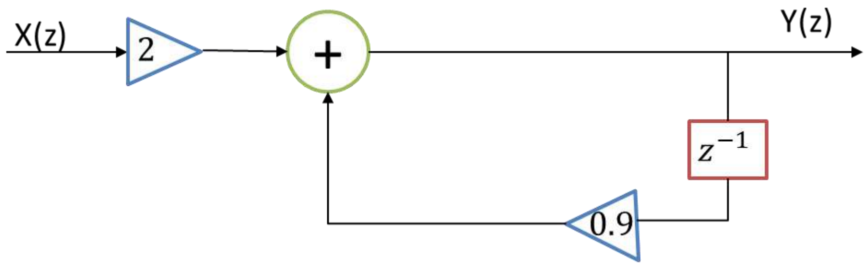


3.4 Frequency Response of LTI Processor :-

Most discrete time signals encountered in practice can be represented as a linear combination of a very large, may be infinite number of sinusoidal discrete time signals of different angular frequencies- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superpositions property of the system. Since a sinusoidal signal can be expressed in terms of an exponential signal, the response of the LTI system to an exponential input is of practical interest- This leads to the concept of frequency response, a transform-domain representation of the LTI discrete time system.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n} = H[k]_{|\frac{2\pi k}{N}=w} = H(z)_{|z=e^{jw}} = H(s)_{|s=jw}$$

Ex: find and sketch the frequency response of the system shown below



Sol:

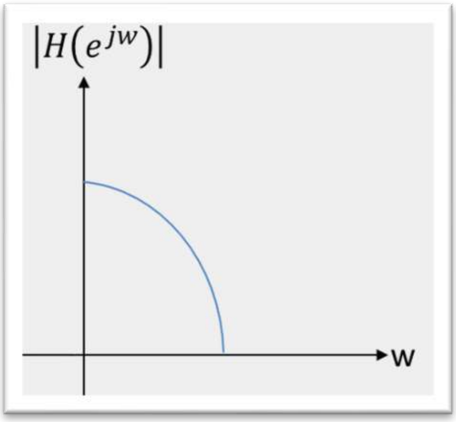
$$Y(z) = 2X(z) + 0.9z^{-1}Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - 0.9z^{-1}}$$

$$freq.response = H(e^{jw}) = H(z)_{|z=e^{jw}}$$

$$\therefore H(e^{jw}) = \frac{2}{1 - 0.9e^{-jw}} = \frac{2}{1 - 0.9\cos w + j0.9\sin w}$$

$$|H(e^{jw})| = \frac{2}{\sqrt[2]{(1 - 0.9\cos w)^2 + (0.9\sin w)^2}}$$



Note: the studying of the frequency response of the systems lead us to determine the behavior of the system with the frequencies.