Adaptive Control: Introduction, Overview, and Applications

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Course Overview

- **Motivating Example**
- **Review of Lyapunov Stability Theory**
  - Nonlinear systems and equilibrium points
  - Linearization
  - Lyapunov's direct method
  - Barbalat's Lemma, Lyapunov-like Lemma, Bounded Stability
- **Model Reference Adaptive Control**
  - Basic concepts
  - 1^{st} order systems
  - n^{th} order systems
  - Robustness to Parametric / Non-Parametric Uncertainties
- **Neural Networks, (NN)**
  - Architectures
  - Using sigmoids
  - Using Radial Basis Functions, (RBF)
- **Adaptive NeuroControl**
- **Design Example: Adaptive Reconfigurable Flight Control using RBF NN-s**

References

- J-J. E. Slotine and W. Li, *Applied Nonlinear Control*, Prentice-Hall, New Jersey, 1991
- S. Haykin, *Neural Networks: A Comprehensive Foundation*, 2nd edition, Prentice-Hall, New Jersey, 1999
- H. K., Khalil, *Nonlinear Systems*, 2nd edition, Prentice-Hall, New Jersey, 2002
- Recent Journal / Conference Publications, (available upon request)

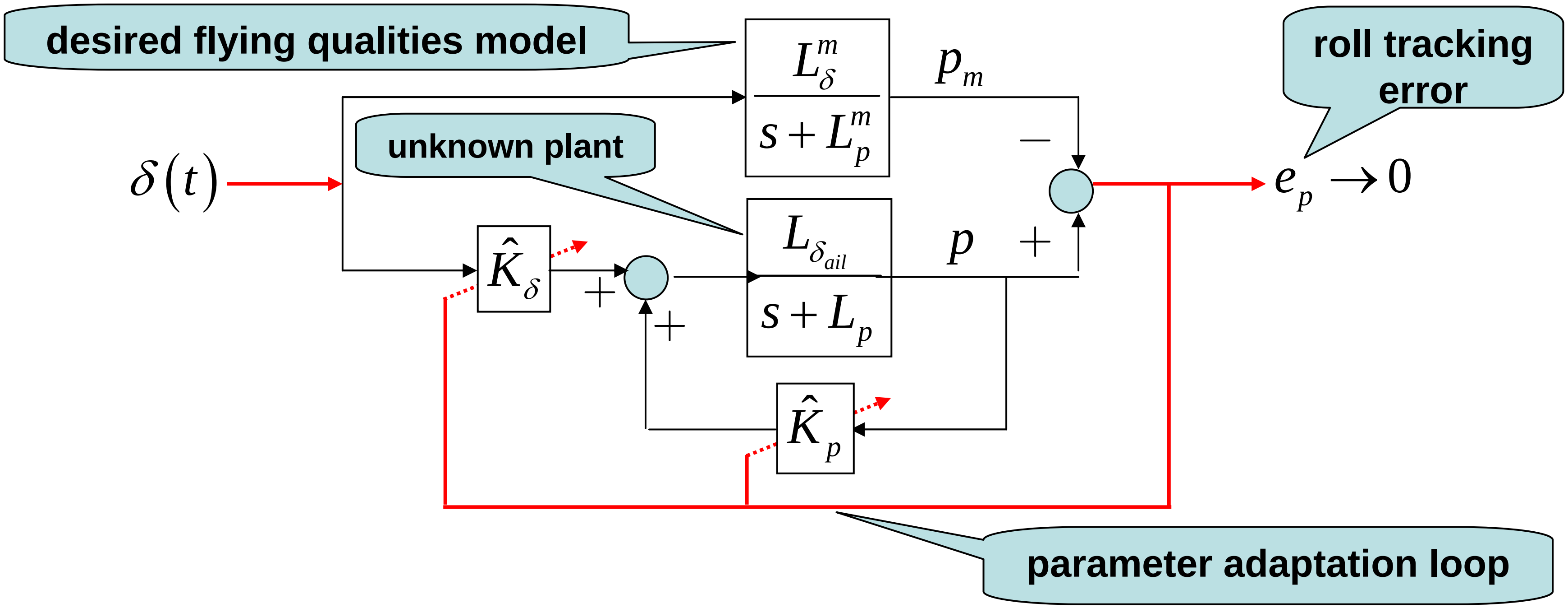
Motivating Example: Roll Dynamics
(Model Reference Adaptive Control)

- ***Uncertain*** **Roll dynamics**: → $\dot{p} = L_p\, p + L_{\delta_{ail}}\, \delta_{ail}$
  - p is roll rate,
  - δ_{ail} is aileron position
  - $\left(L_p,\, L_{\delta_{ail}}\right)$ are *unknown* damping, aileron effectiveness
- **Flying Qualities Model**: → $\dot{p}_m = L_p^m\, p_m + L_\delta^m\, \delta(t)$
  - $\left(L_p^m,\, L_\delta^m\right)$ are *desired* damping, control effectiveness
  - $\delta(t)$ is a reference input, (pilot stick, guidance command)
  - roll rate tracking error: → $e_p(t) = \left(p(t) - p_m(t)\right) \to 0$
- **Adaptive Roll Control**: → $\delta_{ail} = \hat{K}_p\, p + \hat{K}_\delta\, \delta$

$$\begin{cases} \dot{\hat{K}}_p = -\gamma_p\, p\left(p - p_m\right) \\ \dot{\hat{K}}_\delta = -\gamma_{\delta_{ail}}\, \delta(t)\left(p - p_m\right) \end{cases}, \quad \left(\gamma_p, \gamma_{\delta_{ail}}\right) > 0$$

parameter adaptation laws

Motivating Example: Roll Dynamics
(Block-Diagram)



- Adaptive control provides Lyapunov stability
- Design is based on Lyapunov Theorem (2nd method)
- Yields closed-loop asymptotic tracking with all remaining signals bounded in the presence of system uncertainties

Lyapunov Stability Theory

Alexander Michailovich Lyapunov 1857-1918

- Russian mathematician and engineer who laid out the foundation of the Stability Theory
- Results published in 1892, Russia
- Translated into French, 1907
- Reprinted by Princeton University, 1947
- American Control Engineering Community Interest, 1960's

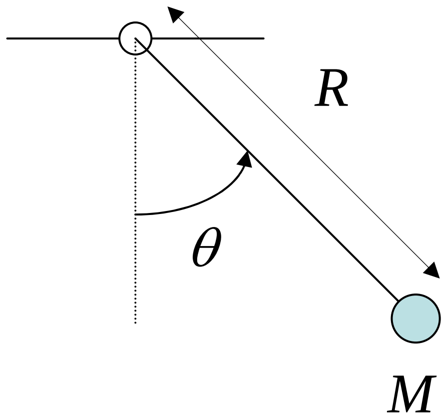
Nonlinear Dynamic Systems and Equilibrium Points

- A nonlinear dynamic system can usually be represented by a set of *n* differential equations in the form: $\dot{x} = f(x,t), \quad \text{with } x \in R^n, t \in R$
  - x is the state of the system
  - t is time
- If f does not depend *explicitly* on time then the system is said to be *autonomous*: $\dot{x} = f(x)$
- A state x_e is an equilibrium if once $x(t) = x_e$, it remains equal to x_e for all future times: $0 = f(x)$

Example: Equilibrium Points of a Pendulum

- System dynamics: $M R^2 \ddot{\theta} + b\dot{\theta} + M g R \sin(\theta) = 0$
- State space representation, $(x_1 = \theta, \quad x_2 = \dot{\theta})$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{b}{M R^2} x_2 - \frac{g}{R}\sin(x_1)
\end{aligned}
$$

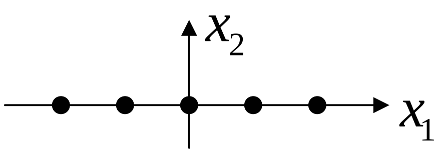


- Equilibrium points:

$$
\begin{aligned}
0 &= x_2 \\
0 &= -\frac{b}{M R^2} x_2 - \frac{g}{R}\sin(x_1)
\end{aligned}
$$

$$x_2 = 0, \quad \sin(x_1) = 0$$

$$x_e = \begin{pmatrix} \pi k \\ 0 \end{pmatrix}, \quad (k = 0, \pm 1, \pm 2, \ldots)$$



Example: Linear Time-Invariant (LTI) Systems

- LTI system dynamics: $\dot{x} = A x$
  - has a single equilibrium point (the origin 0) if *A* is nonsingular
  - has an infinity of equilibrium points in the null-space of *A*: $A x_e = 0$
- LTI system trajectories: $x(t) = \exp\left(A(t - t_0)\right) x(t_0)$
- If *A* has all its eigenvalues in the left half plane then the system trajectories converge to the origin exponentially fast

State Transformation

- Suppose that x_e is an equilibrium point
- Introduce a new variable: $y = x - x_e$
- Substituting for $x = y + x_e$ into $\dot{x} = f(x)$
- New system dynamics: $\dot{y} = f(y + x_e)$
- New equilibrium: $y = 0$, (since $f(x_e) = 0$)
- ***Conclusion***: study the behavior of the new system in the neighborhood of the origin

Nominal Motion

- Let $x^*(t)$ be the solution of $\boxed{\dot{x} = f(x)}$
  - the nominal motion trajectory corresponding to initial conditions $x^*(0) = x_0$
- Perturb the initial condition $\boxed{x(0) = x_0 + \delta x_0}$
- Study the stability of the motion error: $\boxed{e(t) = x(t) - x^*(t)}$

- The error dynamics:
  - non-autonomous!

$$\boxed{\begin{aligned} &\dot{e} = f\left(x^*(t) + e(t)\right) - f\left(x^*(t)\right) = g(e,t) \\ &e(0) = \delta x_0 \end{aligned}}$$

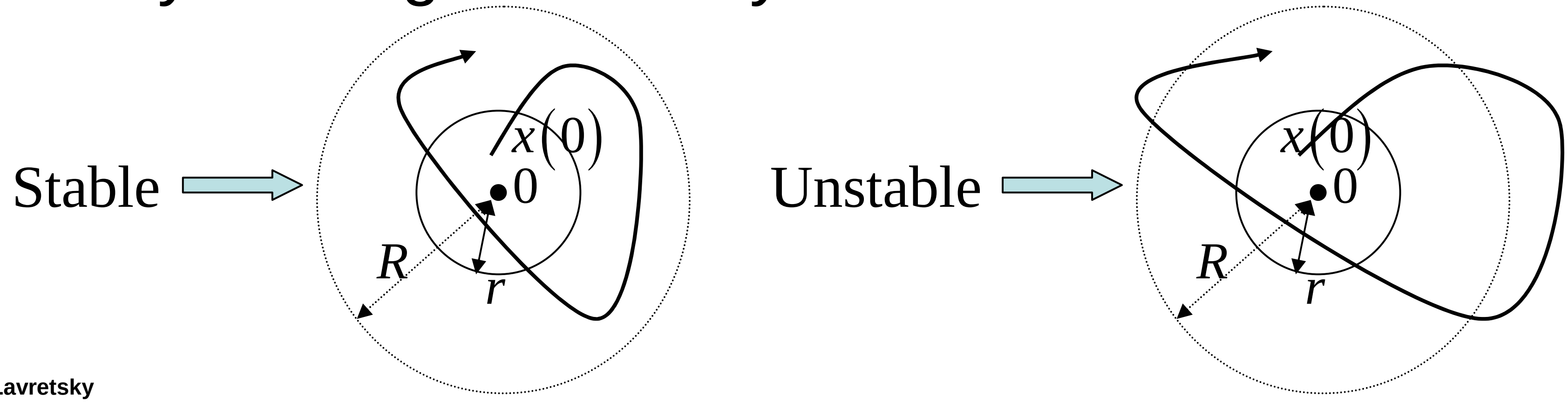
- <u>*Conclusion*</u>: Instead of studying stability of the nominal motion, study stability of the error dynamics w.r.t. the origin

Lyapunov Stability

- **Definition**: The equilibrium state *x* = 0 of autonomous nonlinear dynamic system is said to be *stable* if:

$$\forall R > 0, \quad \exists r > 0, \quad \left\{ \|x(0)\| < r \right\} \Rightarrow \left\{ \forall t \geq 0, \|x(t)\| < R \right\}$$

- Lyapunov Stability means that the system trajectory can be kept arbitrary close to the origin by starting sufficiently close to it



Asymptotic Stability

- **Definition**: An equilibrium point 0 is *asymptotically stable* if it is stable and if in addition: $\exists r > 0, \quad \{\|x(0)\| < r\} \Rightarrow \left\{\lim_{t\to\infty}\|x(t)\| = 0\right\}$
- Asymptotic stability means that the equilibrium is stable, and that in addition, states started close to 0 actually converge to 0 as time *t* goes to infinity
- Equilibrium point that is stable but not asymptotically stable is called *marginally stable*

Exponential Stability

- **Definition**: An equilibrium point 0 is <u>*exponentially stable*</u> if:

$$\exists\, r, \alpha, \lambda > 0, \quad \forall \left\{ \|x(0)\| < r \wedge t > 0 \right\} : \quad \|x(t)\| \le \alpha \|x(0)\| e^{-\lambda t},$$

- The state vector of an exponentially stable system converges to the origin faster than an exponential function
- Exponential stability implies asymptotic stability

Local and Global Stability

- **Definition**: If asymptotic (exponential) stability holds for any initial states, the equilibrium point is called globally asymptotically (exponentially) stable.
- Linear time-invariant (LTI) systems are either exponentially stable, marginally stable, or unstable. Stability is always global.
- Local stability notion is needed only for nonlinear systems.
- **Warning**: State convergence does not imply stability!

Lyapunov's 1st Method

- Consider autonomous nonlinear dynamic system: $\dot{x} = f(x)$
- Assume that $f(x)$ is continuously differentiable
- Perform linearization:

$$\dot{x} = \underbrace{\left(\frac{\partial f(x)}{\partial x}\right)_{x=0}}_{A} x + \underbrace{f_{h.o.t.}(x)}_{\text{higher-order terms}} \cong A x$$

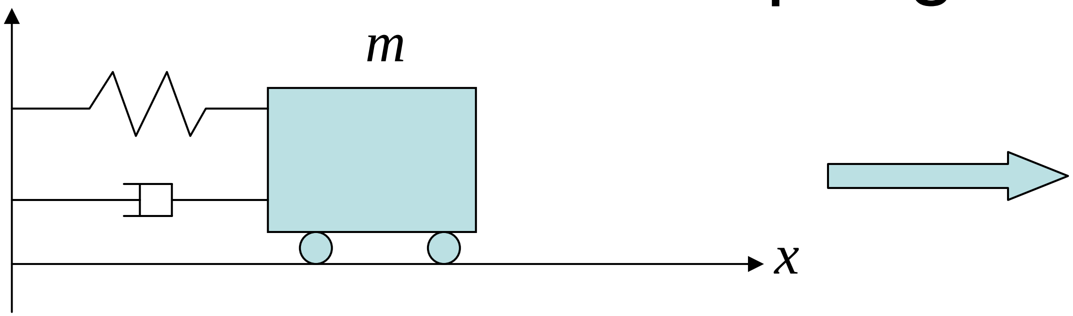
- **Theorem**
  - If A is Hurwitz then the equilibrium is asymptotically stable, (locally!)
  - If A has at least one eigenvalue in right-half complex plane then the equilibrium is unstable
  - If A has at least one eigenvalue on the imaginary axis then one cannot conclude anything from the linear approximation

Lyapunov's Direct (2nd) Method

- **Fundamental Physical Observation**
  – If the total *energy* of a mechanical (or electrical) system is continuously dissipated, then the system, *whether linear or nonlinear*, must eventually settle down to an equilibrium point.
- **Main Idea**
  – Analyze stability of an *n*-dimensional dynamic system by examining the variation of a single *scalar* function, (system energy).

Lyapunov's Direct Method (Motivating Example)

- Nonlinear mass-spring-damper system



$$m\ddot{x} + \underbrace{b\,\dot{x}|\dot{x}|}_{\text{damping}} + \underbrace{k_0\,x + k_1\,x^3}_{\text{spring term}} = 0$$

- **Question**: If the mass is pulled away and then released, will the resulting motion be stable?
  - Stability definitions are hard to verify
  - Linearization method fails, (linear system is only marginally stable

Lyapunov's Direct Method (Motivating Example, continued)

- Total mechanical energy

$$V(x) = \underbrace{\frac{1}{2} m \dot{x}^2}_{\text{kinetic}} + \underbrace{\int_0^x \left(k_0\, x + k_1\, x^3\right) dx}_{\text{potential}} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_0\, x^2 + \frac{1}{4} k_1\, x^4$$

- Total energy rate of change along the system's motion:

$$\dot{V}(x) = m\,\dot{x}\,\ddot{x} + \left(k_0\, x + k_1\, x^3\right)\dot{x} = \dot{x}\left(-b\,\dot{x}\left|\dot{x}\right|\right) = -b\left|\dot{x}\right|^3 \le 0$$

- ***Conclusion***: Energy of the system is dissipated until the mass settles down: $\dot{x} = 0$

Lyapunov's Direct Method (Overview)

- Method
  - based on generalization of energy concepts
- Procedure
  - generate a scalar "energy-like function (*Lyapunov function*) for the dynamic system, and examine its variation in time, (derivative along the system trajectories)
  - if energy is dissipated (derivative of the Lyapunov function is non-positive) then conclusions about system stability may be drawn

Positive Definite Functions

- **Definition**: A scalar continuous function $V(x)$ is called *locally positive definite* if

$$V(0) = 0 \wedge \{\forall x \neq 0 \wedge \|x\| < R\} \Rightarrow V(x) > 0$$

- If $V(0) = 0 \wedge \{\forall x \neq 0\} \Rightarrow V(x) > 0$ then $V(x)$ is *globally positive definite*

- Remarks
  - a positive definite function must have a unique minimum $\min_{x \in B_R} V(x) = V(x_{\min}) = V_{\min}$
  - if $V_{\min}$ =/= 0 or $x_{\min}$ =/= 0 then use

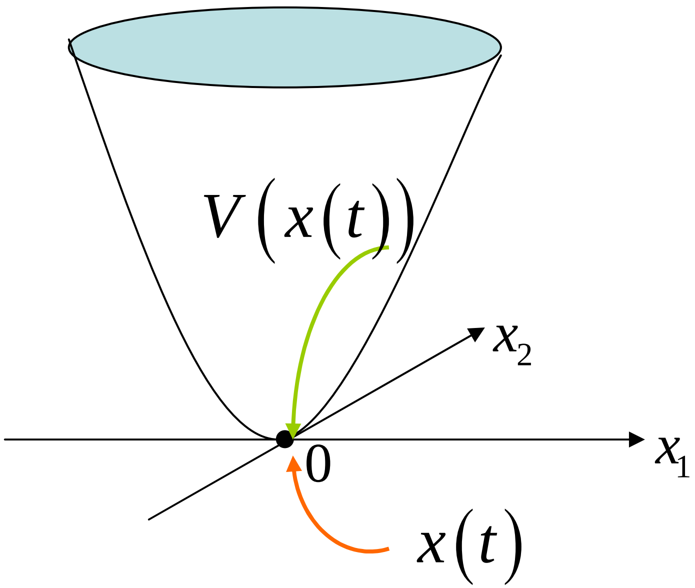
$$W(x) = V(x - x_{\min}) - V_{\min}$$

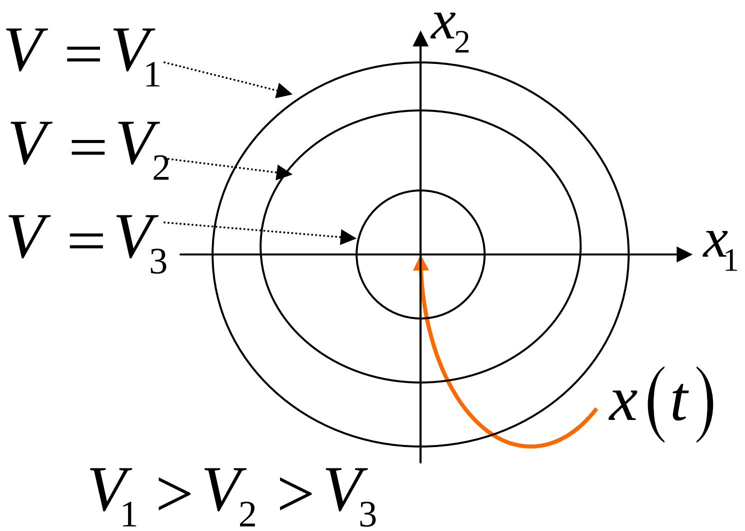
Lyapunov Functions

- **Definition**: If in a ball B_R the function $V(x)$ is positive definite, has continuous partial derivatives, and if its time derivative along any state trajectory of the system $\dot{x} = f(x)$ is negative semi-definite, i.e., $\dot{V}(x) \le 0$ then $V(x)$ is said to be a *Lyapunov function* for the system.
- *Time derivative* of the Lyapunov function

$$\dot{V}(x) = \nabla V(x)\, f(x) \le 0, \quad \nabla V(x) = \begin{pmatrix} \dfrac{\partial V(x)}{\partial x_1} & \dots & \dfrac{\partial V(x)}{\partial x_n} \end{pmatrix} \in R^n$$

Lyapunov Function (Geometric Interpretation)





- Lyapunov function is a bowl, (locally)
- $V(x(t))$ always moves down the bowl
- System state moves across contour curves of the bowl towards the origin

Lyapunov Stability Theorem

- If in a ball B_R there exists a scalar function $V(x)$ with continuous partial derivatives such that $\boxed{\forall x \in B_R : \quad V(x) > 0 \wedge \dot{V}(x) \le 0}$ then the equilibrium point 0 is ***stable***
  - If the time derivative is locally negative definite $\boxed{\dot{V}(x) < 0}$ then the stability is ***asymptotic***
    - If *V*(*x*) is radially unbounded, i.e., $\lim_{\|x\| \to \infty} V(x) = \infty$, then the origin is ***globally asymptotically stable***
- *V*(*x*) is called the Lyapunov function of the system

Example: Local Stability

- Pendulum with viscous damping: $\ddot{\theta} + \dot{\theta} + \sin\theta = 0$
- State vector: $x = \begin{pmatrix} \theta & \dot{\theta} \end{pmatrix}^T$
- Lyapunov function candidate: $V(x) = (1 - \cos\theta) + \dfrac{\dot{\theta}^2}{2}$
  - represents the total energy of the pendulum
  - locally positive definite
  - time-derivative is *negative semi-definite*

$$\dot{V}(x) = \frac{\partial V(x)}{\partial \theta}\dot{\theta} + \frac{\partial V(x)}{\partial \dot{\theta}}\ddot{\theta} = \dot{\theta}\sin\theta + \dot{\theta}\underbrace{\ddot{\theta}}_{-\dot{\theta}-\sin\theta} = -\dot{\theta}^2 \le 0$$

- ***Conclusion***: System is locally stable

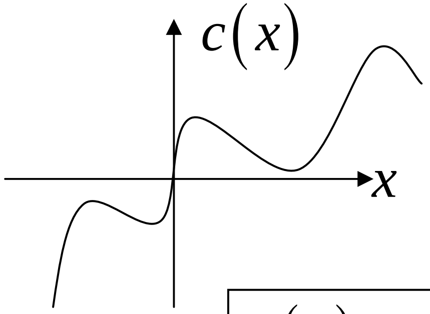
Example: Asymptotic Stability

- System Dynamics: $$\begin{aligned}\dot{x}_1 &= x_1\left(x_1^2 + x_2^2 - 2\right) - 4\,x_1\,x_2^2 \\ \dot{x}_2 &= x_2\left(x_1^2 + x_2^2 - 2\right) + 4\,x_1^2\,x_2\end{aligned}$$
- Lyapunov function candidate: $V\left(x_1, x_2\right) = x_1^2 + x_2^2$
  - positive definite
  - time-derivative is *negative definite* in the 2-dimensional ball defined by $x_1^2 + x_2^2 < 2$

$$\dot{V}\left(x_1, x_2\right) = 2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 2\right) < 0$$

- *Conclusion*: System is *locally* asymptotically stable

Example: Global Asymptotic Stability



- Nonlinear 1st order system

$$\dot{x} = -c(x), \quad \text{where: } x\,c(x) > 0$$

- Lyapunov function candidate:

$$V(x) = x^2$$

  - globally positive definite
  - time-derivative is negative definite

$$\dot{V}(x) = 2\,x\,\dot{x} = -2\,x\,c(x) < 0$$

- *Conclusion*: System is globally asymptotically stable
- **Remark**: Trajectories of a 1st order system are monotonic functions of time, (why?)

La Salle's Invariant Set Theorems

- It often happens that the time-derivative of the Lyapunov function is only negative *semi*-definite
- It is still possible to draw conclusions on the *asymptotic* stability
- Invariant Set Theorems (attributed to La Salle) extend the concept of Lyapunov function

Example: 2nd Order Nonlinear System

- System dynamics: $\ddot{x} + b(\dot{x}) + c(x) = 0$
  - where $b(x)$ and $c(x)$ are continuous functions verifying the sign conditions:
  $$\dot{x}\, b(\dot{x}) > 0, \text{ for } \dot{x} \neq 0$$
  $$x\, c(x) > 0, \text{ for } x \neq 0$$
- Lyapunov function candidate: $V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \int_0^x c(y)\, dy$
  - positive definite
  - time-derivative is negative *semi*-definite
  $$\dot{V} = \dot{x}\,\ddot{x} + c(x)\,\dot{x} = -\dot{x}\, b(\dot{x}) \leq 0$$
    - system energy is dissipated
    $$\dot{x}\, b(\dot{x}) = 0 \Leftrightarrow \dot{x} = 0 \Rightarrow \ddot{x} = -c(x) \Rightarrow x_e = 0$$
    - system cannot get “stuck” at a non-zero equilibrium
- ***Conclusion***: Origin is globally asymptotically stable

Lyapunov Functions for LTI Systems

- LTI system dynamics: $\dot{x} = A\,x$
- Lyapunov function candidate: $V(x) = x^T P\,x$
  - where *P* is symmetric positive definite matrix
  - function V(x) is positive definite
- Time-derivative of *V*(*x*(*t*)) along the system trajectories: $\dot{V}(x) = \dot{x}^T P\,x + x^T P\,\dot{x} = x^T \underbrace{\left(A^T P + P\,A\right)}_{-Q} x = -x^T Q\,x < 0$
  - where Q is symmetric positive definite matrix
  - Lyapunov equation: $A^T P + P\,A = -Q$
- Stability analysis procedure:
  - choose a symmetric positive definite Q
  - solve the Lyapunov equation for P
  - check whether P is positive definite

Stability of LTI Systems

- **Theorem**
  - An LTI system is stable (globally exponentially) if and only if for any symmetric positive definite matrix Q, the unique matrix solution P of the Lyapunov equation is symmetric and positive definite
- **Remark**: In most practical cases Q is chosen to be a diagonal matrix with *positive* diagonal elements

Barbalat's Lemma: Preliminaries

- Invariant set theorems of La Salle provide asymptotic stability analysis tools for *autonomous* systems with a negative *semi*-definite time-derivative of a Lyapunov function
- Barbalat's Lemma extends Lyapunov stability analysis to *non-autonomous* systems, (such as adaptive model reference control)

Barbalat's Lemma

- **Lemma**
  - If a differentiable function *f*(*t*) has a finite limit as $t \to \infty$ and if $\dot{f}(t)$ is ***uniformly continuous***, then $\lim_{t\to\infty} \dot{f}(t) = 0$
- **Remarks**
  - *uniform continuity* of a function is difficult to verify directly
  - simple *sufficient condition*:
    - if derivative is bounded then function is uniformly continuous
  - The fact that derivative goes to zero does not imply that the function has a limit, as t tends to infinity. The converse is also not true, (in general)
  - Uniform continuity condition is very important!

Example: LTI System

- **Statement**: Output of a stable LTI system is uniformly continuous in time
  - System dynamics: $\dot{x} = A\,x + B\,u$
  - Control input *u* is bounded
  - System output: $y = C\,x$
- **Proof**: Since *u* is bounded and the system is stable then *x* is bounded. Consequently, the output time-derivative $\dot{y} = C\,\dot{x} = C\left(A\,x + B\,u\right)$ is bounded. Thus, (using Barbalat's Lemma), we conclude that the output *y* is ***uniformly continuous*** in time.

Lyapunov-Like Lemma

- If a scalar function $V(x,t)$ satisfies the following conditions
  - function is lower bounded
  - its time-derivative along the system trajectories is negative semi-definite and uniformly continuous in time
- Then: $\lim_{t \to \infty} \dot{V}(x,t) = 0$
- **Question**: Why is this fact so important?
- **Answer**: It provides *theoretical* foundations for *stable* adaptive control design

Example: Stable Adaptation

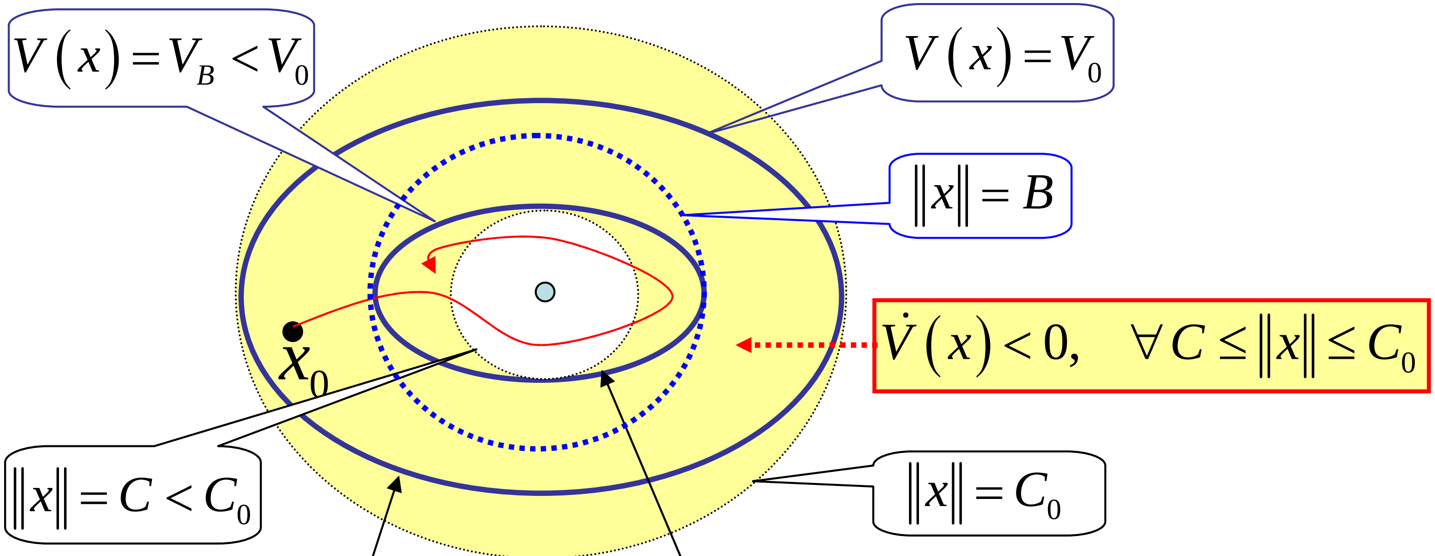
- Closed-loop error dynamics of an adaptive system $\dot{e} = -e + \theta\, w(t),\ \dot{\theta} = -e\, w(t)$
  - where *e* is the tracking error, θ is the parameter error, and *w*(*t*) is a bounded continuous function
- Stability Analysis
  - Consider Lyapunov function candidate: $V(e,\theta) = e^2 + \theta^2$
    - it is positive definite
    - its time-derivative is negative semi-definite $\dot{V}(e,\theta) = 2e(-e + \theta w) + 2\theta(-e w) = -2e^2 \le 0$
    - consequently, *e* and θ are bounded
    - since $\ddot{V}(e,\theta) = -4e(-e + \theta w)$ is bounded, $\dot{V}(e,\theta)$ is uniformly continuous
    - hence: $\lim_{t\to\infty}\left(-2e^2\right) = \lim_{t\to\infty}\dot{V}(e,\theta) = 0 \Rightarrow \lim_{t\to\infty} e(t)$

Uniform Ultimate Boundedness

- **Definition**: The solutions of $\dot{x} = f(x,t)$ starting at $x(t_0) = x_0$ are Uniformly Ultimately Bounded (UUB) with ultimate bound *B* if:

$$\exists C_0 > 0,\ T = T(C_0, B) > 0: \quad \left(\|x(t_0)\| \le C_0\right) \Rightarrow \left(\|x(t)\| \le B, \quad \forall t \ge t_0 + T\right)$$

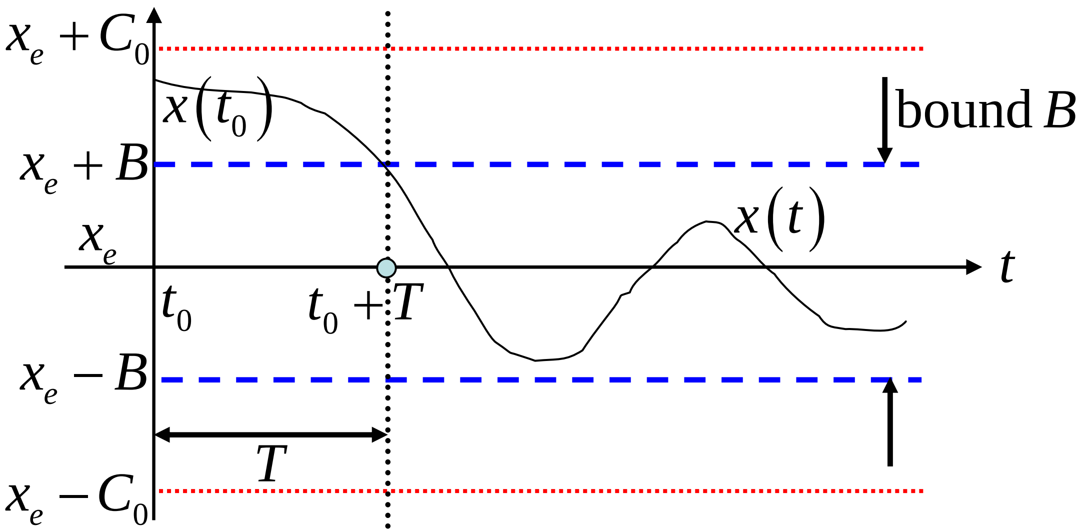
- Lyapunov analysis can be used to show UUB



All trajectories starting in large ellipse enter small ellipse within finite time $T(C_0, B)$

UUB Example : 1st Order System

- The equilibrium point x_e is UUB if there exists a constant C_0 such that for every initial state $x(t_0)$ in an interval $|x(t_0)| \le C_0$ there exists a bound B and a time $T(B, x(t_0))$ such that $|x(t) - x_e| \le B$ for all $t \ge t_0 + T$



UUB by Lyapunov Extension

- Milder form of stability than SISL
- More useful for controller design in practical systems with unknown bounded disturbances:

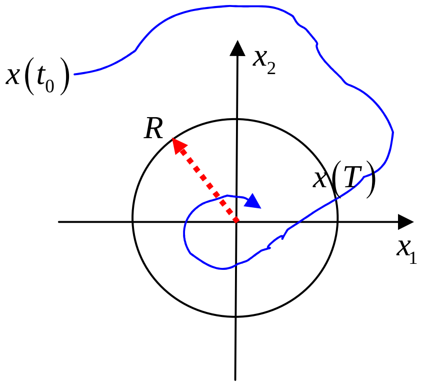
$$\dot{x} = f(x) + d(x)$$

- <u>*Theorem*</u>: Suppose that there exists a function *V*(*x*) with continuous partial derivatives such that for x in a compact set $S \subset R^n$
  - *V*(*x*) is positive definite: $V(x) > 0, \quad \forall \|x\| \neq 0$
  - time derivative of *V*(*x*) is negative definite <u>outside of *S*:</u> $\dot{V}(x) < 0, \quad \forall \|x\| > R, \quad \left(\|x\| \le R\right) \Rightarrow \left(x \in S\right)$
  - Then the system is UUB and $\|x(t)\| \le R, \quad \forall t \ge t_0 + T$

Example: UUB by Lyapunov Extension

- System:

$$\begin{aligned}\dot{x}_1 &= x_1\, x_2^2 - x_1\left(x_1^2 + x_2^2 - 9\right)\\ \dot{x}_2 &= -x_1^2\, x_2 - x_2\left(x_1^2 + x_2^2 - 9\right)\end{aligned}$$



- Lyapunov function candidate:

$$V(x_1, x_2) = x_1^2 + x_2^2$$

- Time derivative:

$$\dot{V}(x_1, x_2) = 2\left(x_1\,\dot{x}_1 + x_2\,\dot{x}_2\right) = -2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 9\right)$$

- Time derivative negative outside compact set

$$\dot{V}(x_1, x_2) < 0, \quad \forall \left\{x : x_1^2 + x_2^2 > 9\right\}$$

- ***Conclusion***: All trajectories enter circle of radius *R* = 3, in a finite time

Adaptive Control

Introduction

- Basic Ideas in Adaptive Control
  - estimate uncertain plant / controller parameters on-line, while using measured system signals
  - use estimated parameters in control input computation
- Adaptive controller is a dynamic system with on-line parameter estimation
  - inherently nonlinear
  - analysis and design rely on the Lyapunov Stability Theory

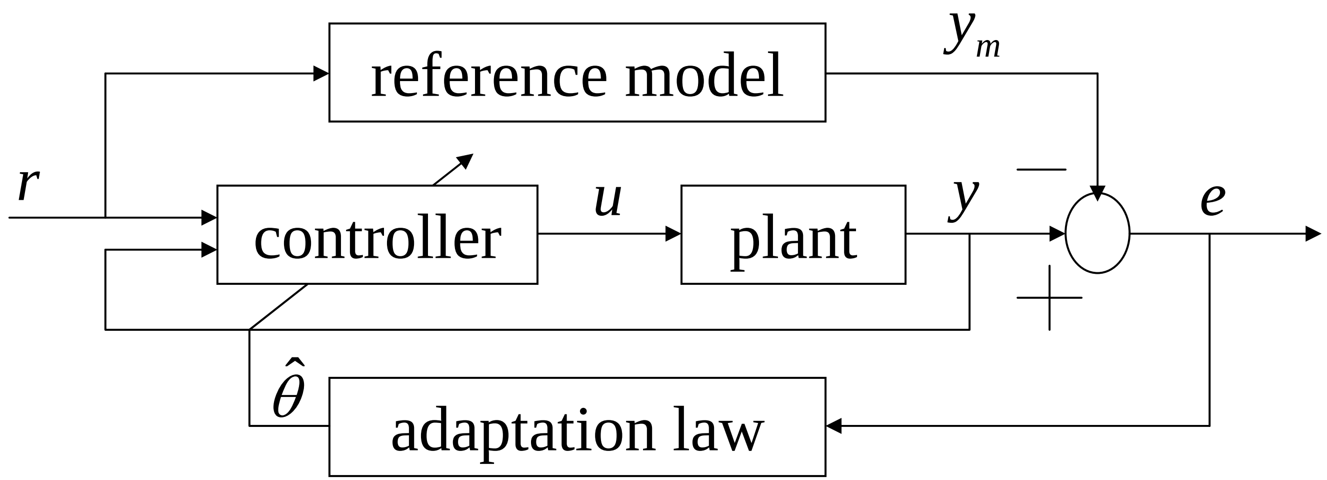
Historical Perspective

- Research in adaptive control started in the early 1950's
  - autopilot design for high-performance aircraft
- Interest diminished due to the crash of a test flight
  - Question: X-?? aircraft tested
- Last decade witnessed the development of a coherent theory and many practical applications

Concepts

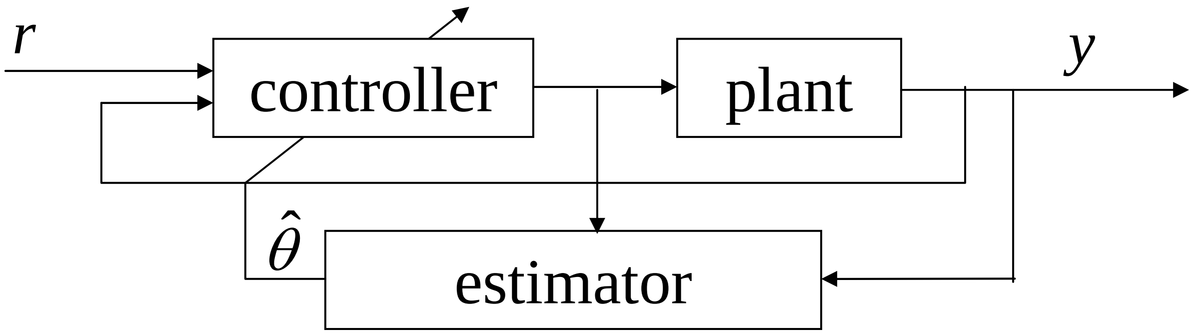
- **Why Adaptive Control?**
  - dealing with complex systems that have unpredictable parameter deviations and uncertainties
- **Basic Objective**
  - maintain consistent performance of a system in the presence of uncertainty and variations in plant parameters
- Adaptive control is superior to robust control in dealing with uncertainties in constant or slow-varying parameters
- Robust control has advantages in dealing with disturbances, quickly varying parameters, and unmodeled dynamics
- ***Solution***: Adaptive augmentation of a Robust Baseline controller

Model-Reference Adaptive Control (MRAC)



- *Plant* has a known structure but the parameters are unknown
- *Reference model* specifies the ideal (desired) response y_m to the external command r
- *Controller* is parameterized and provides tracking
- *Adaptation* is used to adjust parameters in the control law

Self-Tuning Controllers (STC)



- Combines a controller with an on-line (recursive) plant parameter estimator
- Reference model can be added
- Performs simultaneous parameter identification and control
- Uses *Certainty Equivalence Principle*
  - controller parameters are computed from the estimates of the plant parameters as if they were the true ones

Direct vs. Indirect Adaptive Control

- Indirect
  - estimate plant parameters
  - compute controller parameters
  - relies on convergence of the estimated parameters to their true unknown values
- Direct
  - no plant parameter estimation
  - estimate controller parameters (gains) only
- MRAC and STC can be designed using both Direct and Indirect approaches
- *We consider Direct MRAC design*

MRAC Design of 1st Order Systems

- System Dynamics: $\dot{x} = a\,x + b\left(u - f(x)\right)$
  - a, b are constant ***unknown*** parameters
  - ***uncertain*** nonlinear function: $f(x) = \sum_{i=1}^{N} \theta_i\, \varphi_i(x) = \theta^T\, \Phi(x)$
    - vector of constant ***unknown*** parameters: $\theta = \left(\theta_1 \quad \ldots \quad \theta_N\right)^T$
    - vector of known basis functions: $\Phi(x) = \left(\varphi_1(x) \quad \ldots \quad \varphi_N(x)\right)^T$
- Stable Reference Model: $\dot{x}_m = a_m\, x_m + b_m\, r, \quad (a_m < 0)$
- **Control Goal**
  - find *u* such that: $\lim_{t\to\infty}\left(x(t) - x_m(t)\right) = 0$

MRAC Design of 1st Order Systems (continued)

- Control Feedback: $u = \hat{k}_x\, x + \hat{k}_r\, r + \hat{\theta}^T\, \Phi(x)$
  - (N + 2) parameters to estimate on-line: $\hat{k}_x, \hat{k}_r, \hat{\theta}$
- Closed-Loop System: $\dot{x} = \left(a + b\,\hat{k}_x\right) x + b\left(\hat{k}_r\, r + \left(\hat{\theta} - \theta\right)^T \Phi(x)\right)$
- Desired Dynamics: $\dot{x}_m = a_m\, x_m + b_m\, r$
- Matching Conditions Assumption
  - there exist *ideal* gains (k_x, k_r) such that: $\begin{aligned} a + b\,k_x &= a_m \\ b\,k_r &= b_m \end{aligned}$
  - *Note*: knowledge of the ideal gains is not required, only their existence is needed
  - consequently:

$$a + b\,\hat{k}_x - a_m = a + b\,\hat{k}_x - a - b\,k_x = b\left(\hat{k}_x - k_x\right) = b\,\Delta k_x$$

$$b\,\hat{k}_r - b_m = b\,\hat{k}_r - b\,k_r = b\left(\hat{k}_r - k_r\right) = b\,\Delta k_r$$

MRAC Design of 1st Order Systems (continued)

- Tracking Error: $e(t) = x(t) - x_m(t)$
- Error Dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) = \left(a + b\,\hat{k}_x\right)x + b\left(\hat{k}_r\,r + \underbrace{\left(\hat{\theta} - \theta\right)^T}_{\Delta\theta}\Phi(x)\right) - a_m\,x_m - b_m\,r \pm a_m\,x$$

$$= a_m\left(x - x_m\right) + \left(a + b\,\hat{k}_x - a_m\right)x + b\left(\hat{k}_r - k_r\right)r + b\,\Delta\theta^T\,\Phi(x)$$

$$= a_m\,e + b\left(\Delta k_x\,x + \Delta k_r\,r + \Delta\theta^T\,\Phi(x)\right)$$

- Lyapunov Function Candidate:

$$V\left(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\theta(t)\right) = e^2 + |b|\left(\gamma_x^{-1}\,\Delta k_x^2 + \gamma_r^{-1}\,\Delta k_r^2 + \Delta\theta^T\,\Gamma_\theta^{-1}\,\Delta\theta\right)$$

  - where: $\gamma_x > 0, \gamma_r > 0, \text{and } \Gamma = \Gamma^T > 0$ is symmetric positive definite matrix

MRAC Design of 1st Order Systems (continued)

- Time-derivative of the Lyapunov function

$$
\begin{aligned}
\dot{V}(e, \Delta k_x, \Delta k_r, \Delta\theta) &= 2e\dot{e} + 2|b|\left(\gamma_x^{-1}\Delta k_x \dot{\hat{k}}_x + \gamma_r^{-1}\Delta k_r \dot{\hat{k}}_r + \Delta\theta^T \Gamma_\theta^{-1}\dot{\hat{\theta}}\right) \\
&= 2e\left(a_m e + b\left(\Delta k_x x + \Delta k_r r\right) + \Delta\theta^T \Phi(x)\right) \\
&+2|b|\left(\gamma_x^{-1}\Delta k_x \dot{\hat{k}}_x + \gamma_r^{-1}\Delta k_r \dot{\hat{k}}_r + \Delta\theta^T \Gamma_\theta^{-1}\dot{\hat{\theta}}\right) \\
&= 2a_m e^2 + 2|b|\left(\Delta k_x\left(x e \operatorname{sgn}(b) + \gamma_x^{-1}\dot{\hat{k}}_x\right)\right) \\
&+2|b|\left(\Delta k_r\left(r e \operatorname{sgn}(b) + \gamma_r^{-1}\dot{\hat{k}}_r\right)\right) + 2|b|\Delta\theta^T\left(\Phi(x) e \operatorname{sgn}(b) + \Gamma_\theta^{-1}\dot{\hat{\theta}}\right)
\end{aligned}
$$

MRAC Design of 1st Order Systems (continued)

- Adaptive Control Design Idea
  - Choose adaptive laws, (on-line parameter updates) such that the time-derivative of the Lyapunov function decreases along the error dynamics trajectories

$$
\begin{aligned}
\dot{\hat{k}}_x &= -\gamma_x \, x \, e \, \mathrm{sgn}(b) \\
\dot{\hat{k}}_r &= -\gamma_r \, r \, e \, \mathrm{sgn}(b) \\
\dot{\hat{\theta}} &= -\Gamma_\theta \, \Phi(x) \, e \, \mathrm{sgn}(b)
\end{aligned}
$$

- Time-derivative of the Lyapunov function becomes semi-negative definite!

$$
\dot{V}\left(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\theta(t)\right) = 2 \underbrace{a_m}_{<0} e(t)^2 \le 0
$$

MRAC Design of 1st Order Systems (continued)

- Closed-Loop System Stability Analysis
  - Since $V \geq 0 \text{ and } \dot{V} \leq 0$ then all the parameter estimation errors are bounded
  - Since the true (unknown) parameters are constant then all the estimated parameters are bounded
- Assumption
  - reference input r(t) is bounded
- Consequently, x_m and $\dot{x}_m$ are bounded

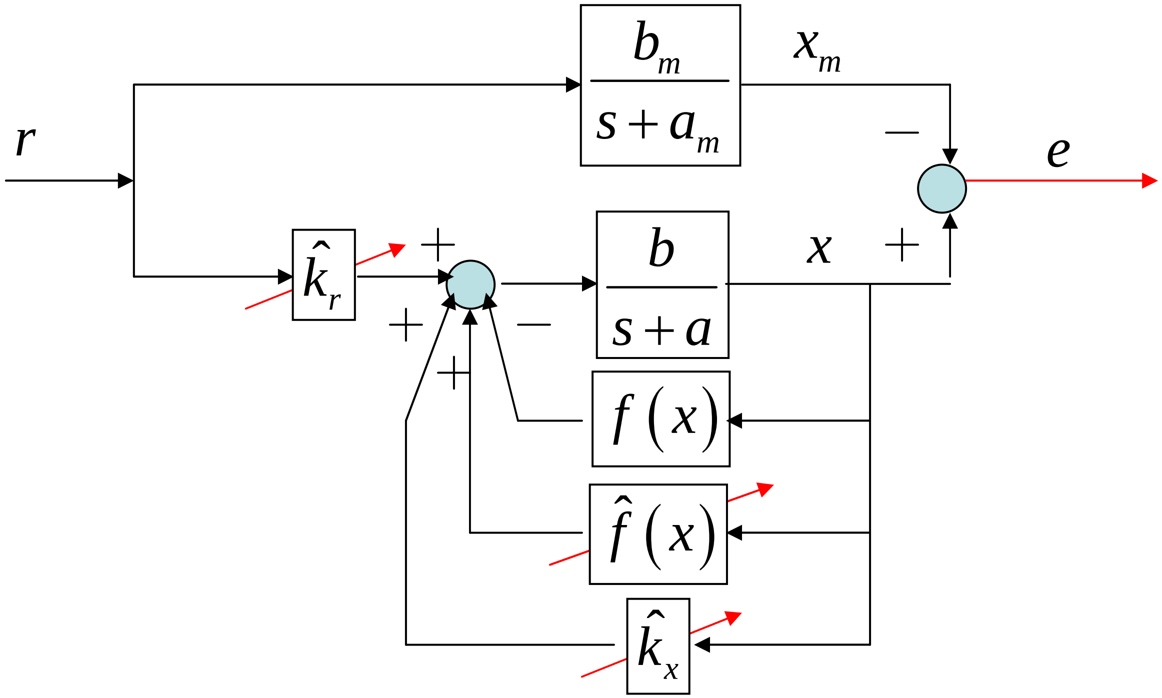
MRAC Design of 1st Order Systems (continued)

- Since $x = e + x_m$ then x is bounded
- Consequently, the adaptive control feedback *u* is bounded
- Thus, $\dot{x}$ is bounded, and $\dot{e} = \dot{x} - \dot{x}_m$ is bounded, as well
- It immediately follows that $\boxed{\ddot{V} = 4\,a_m\,e(t)\,\dot{e}(t)}$ is bounded
- Using Barbalat's Lemma we conclude that $\dot{V}(t)$ is uniformly continuous function of time

MRAC Design of 1st Order Systems (completed)

- Using Lyapunov-like Lemma: $\lim_{t \to \infty} \dot{V}(x, t) = 0$
- Since $\dot{V} = 2 a_m e(t)^2$ it follows that: $\lim_{t \to \infty} e(t) = 0$
- **Conclusions**
  - achieved asymptotic tracking: $x(t) \to x_m(t), \text{ as } t \to \infty$
  - all signals in the closed-loop system are bounded

MRAC Design of 1st Order Systems (Block-Diagram)



- Adaptive gains: $\hat{k}_x(t), \hat{k}_r(t)$
- On-line function estimation: $\hat{f}(x) = \hat{\theta}^T(t)\Phi(x) = \sum_{i=1}^{N} \hat{\theta}_i(t)\varphi_i(x)$

Adaptive Dynamic Inversion (ADI) Control

ADI Design of 1st Order Systems

- System Dynamics: $\dot{x} = a\,x + b\,u + f(x)$
  - a, b are constant ***unknown*** parameters
  - ***uncertain*** nonlinear function: $f(x) = \sum_{i=1}^{N} \theta_i\,\varphi_i(x) = \theta^T\,\Phi(x)$
    - vector of constant ***unknown*** parameters: $\theta = \begin{pmatrix} \theta_1 & \ldots & \theta_N \end{pmatrix}^T$
    - vector of known basis functions: $\Phi(x) = \begin{pmatrix} \varphi_1(x) & \ldots & \varphi_N(x) \end{pmatrix}^T$
- Stable Reference Model: $\dot{x}_m = a_m\,x_m + b_m\,r, \quad (a_m < 0)$
- **Control Goal**
  - find *u* such that: $\lim_{t\to\infty}\left(x(t) - x_m(t)\right) = 0$

ADI Design of 1st Order Systems (continued)

- Rewrite system dynamics:

$$\dot{x} = \hat{a}\,x + \hat{b}\,u + \hat{f}(x) - \underbrace{(\hat{a} - a)}_{\Delta a}x - \underbrace{(\hat{b} - b)}_{\Delta b}u - \underbrace{\left(\hat{f}(x) - f(x)\right)}_{\Delta f(x)}$$

- Function estimation error:

$$\Delta f(x) \triangleq \hat{f}(x) - f(x) = \underbrace{(\hat{\theta} - \theta)}_{\Delta\theta}^{T}\Phi(x)$$

- On-line estimated parameters: $\hat{a},\quad \hat{b},\quad \hat{\theta}$
- Parameter estimation errors

$$\Delta a \triangleq \hat{a} - a,\quad \Delta b \triangleq \hat{b} - b,\quad \Delta\theta \triangleq \hat{\theta} - \theta$$

ADI Design of 1st Order Systems (continued)

- ADI Control Feedback: $u = \frac{1}{\hat{b}}\left(\left(a_m - \hat{a}\right)x + b_m\, r\right) - \hat{\theta}^T\, \Phi\left(x\right)$
  - $(N + 2)$ parameters to estimate on-line: $\hat{a}, \hat{b}, \hat{\theta}$
  - Need to protect $\hat{b}$ from crossing zero
- Closed-Loop System: $\dot{x} = a_m\, x + b_m\, r - \Delta a\, x - \Delta b\, u - \Delta\theta\, \Phi\left(x\right)$
- Desired Dynamics: $\dot{x}_m = a_m\, x_m + b_m\, r$
- Tracking error: $e \triangleq x - x_m$
- Tracking error dynamics: $\dot{e} = a_m\, e - \Delta a\, x - \Delta b\, u - \Delta\theta\, \Phi\left(x\right)$
- Lyapunov function candidate

$$V\left(e(t), \Delta a(t), \Delta b(t), \Delta\theta(t)\right) = e^2 + \gamma_a^{-1}\,\Delta a^2 + \gamma_b^{-1}\,\Delta b^2 + \Delta\theta^T\, \Gamma_\theta^{-1}\, \Delta\theta$$

ADI Design of 1st Order Systems (continued)

- Time-derivative of the Lyapunov function

$$\dot{V}\left(e,\Delta a,\Delta b,\Delta\theta\right)=2\,e\,\dot{e}+2\left(\gamma_a^{-1}\,\Delta a\,\dot{\hat{a}}+\gamma_b^{-1}\,\Delta b\,\dot{\hat{b}}+\Delta\theta^T\,\Gamma_\theta^{-1}\,\dot{\hat{\theta}}\right)$$
$$=2\,e\left(a_m\,e-\Delta a\,x-\Delta b\,u-\Delta\theta\,\Phi\left(x\right)\right)$$
$$+2\left(\gamma_a^{-1}\,\Delta a\,\dot{\hat{a}}+\gamma_b^{-1}\,\Delta b\,\dot{\hat{b}}+\Delta\theta^T\,\Gamma_\theta^{-1}\,\dot{\hat{\theta}}\right)$$
$$=2\,a_m\,e^2+\Delta a\left(\gamma_a^{-1}\,\dot{\hat{a}}-x\,e\right)+\Delta b\left(\gamma_b^{-1}\,\dot{\hat{b}}-u\,e\right)+\Delta\theta^T\left(\Gamma_\theta^{-1}\,\dot{\hat{\theta}}-\Phi\left(x\right)e\right)$$

- Adaptive laws

$$\dot{\hat{a}}=\gamma_a\,x\,e$$
$$\dot{\hat{b}}=\gamma_b\,u\,e$$
$$\dot{\hat{\theta}}=\Gamma_\theta\,\Phi\left(x\right)e$$

$$\Rightarrow\quad \dot{V}\left(e,\Delta a,\Delta b,\Delta\theta\right)=2\,a_m\,e^2\le 0$$

System energy decreases

ADI Design of 1st Order Systems (stability analysis)

- Similar to MRAC
- Using Barbalat's Lemma and Lyapunov-like Lemma: $\lim_{t\to\infty} \dot{V}(x,t) = \lim_{t\to\infty}\left[2\,a_m\,e(t)^2\right] = 0$
- Consequently: $\lim_{t\to\infty} e(t) = 0$ ⟹ $x(t) \to x_m(t), \text{ as } t \to \infty$
- **Conclusions**
  - asymptotic tracking
  - all signals in the closed-loop system are bounded

Parameter Convergence ?

- Convergence of adaptive (on-line estimated) parameters to their true unknown values depends on the reference signal $r(t)$
- If $r(t)$ is very simply, (zero or constant), it is possible to have non-ideal controller parameters that would drive the tracking error to zero
- Need conditions for parameter convergence

Persistency of Excitation (PE)

- Tracking error dynamics is a stable filter

$$\dot{e}(t) = a_m\, e + b\underbrace{\left(\Delta k_x\, x + \Delta k_r\, r + \Delta\theta^T\, \Phi(x)\right)}_{\text{Input}}$$

- Since the filter input signal is uniformly continuous and the tracking error asymptotically converges to zero, then when time *t* is large: $\Delta k_x\, x + \Delta k_r\, r + \Delta\theta^T\, \Phi(x) \cong 0$
- Using vector form:

$$\begin{pmatrix} x & r & \Phi^T(x) \end{pmatrix} \begin{pmatrix} \Delta k_x \\ \Delta k_r \\ \Delta\theta \end{pmatrix} \cong 0$$

Persistency of Excitation (PE) (completed)

- If *r*(*t*) is such that $v = \begin{pmatrix} x & r & \Phi^T(x) \end{pmatrix}^T$ satisfies the so-called "*persistent excitation*" conditions, then the adaptive parameter convergence will take place
  - PE Condition: $\exists \alpha > 0 \quad \forall t \quad \exists T > 0 \quad \int_t^{t+T} v(\tau) v^T(\tau) d\tau > \alpha I_{N+2}$
- PE Condition implies that parameter errors converge to zero
  - for linear systems: *m* - sinusoids ensure convergence of (2 *m*) - parameters
  - not known for nonlinear systems

ADI vs. MRAC

- No knowledge about $\operatorname{sgn} b$
- Adaptive laws are similar
- Both methods yield asymptotic tracking that does not rely on Persistency of Excitation (PE) conditions
- ADI needs protection against $\hat{b}$ crossing zero
  - If PE takes place and initial parameter $\hat{b}(0)$ has wrong sign then a control singularity may occur
- Regressor vector $\Phi(x)$ must have ***bounded*** components, (needed for stability proof)

Example: MRAC of a 1st-Order *Linear* System

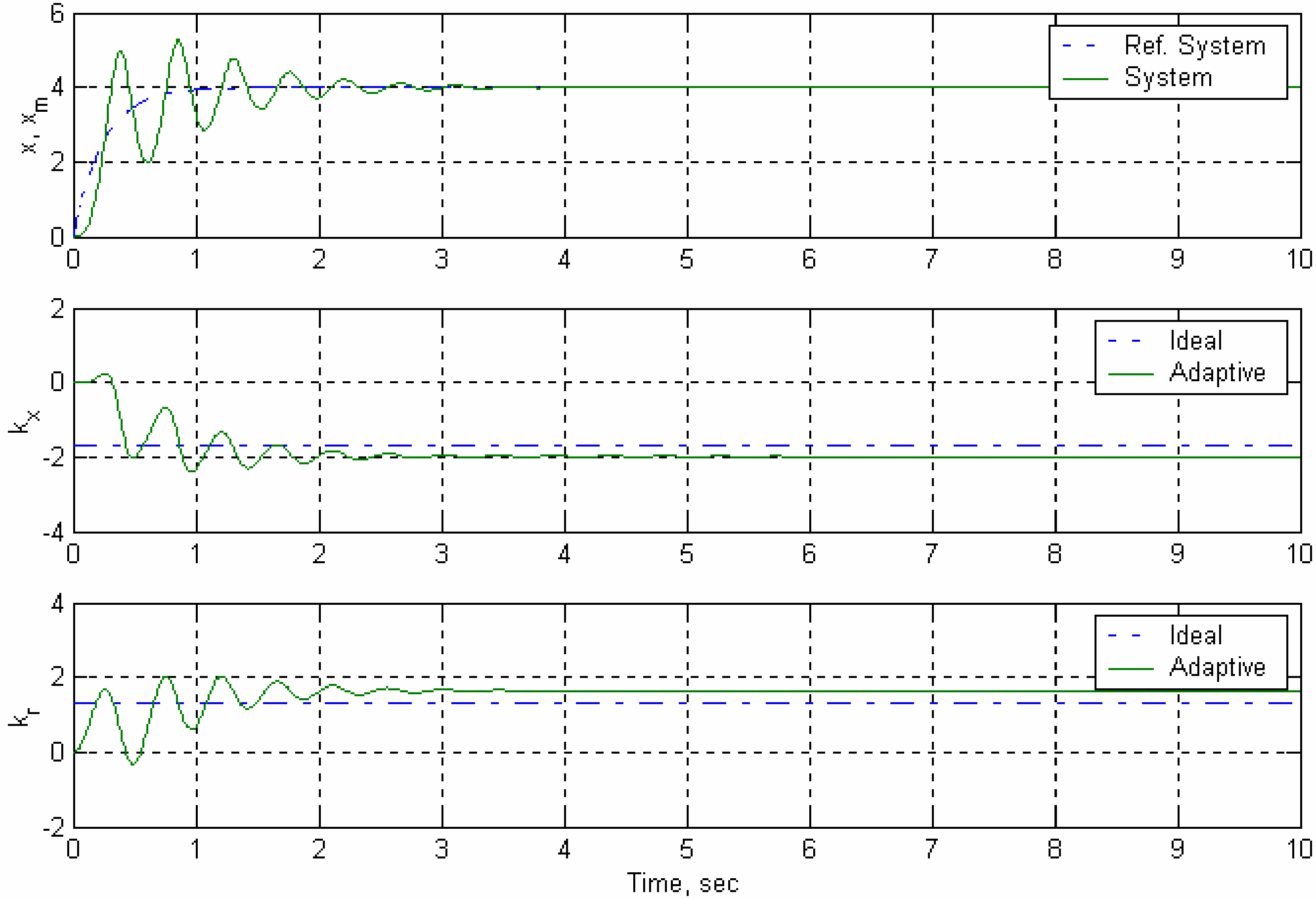
- Unstable Dynamics: $\dot{x} = x + 3u, \quad x(0) = 0$
  - plant parameters $a = 1, \quad b = 3$ are *unknown* to the adaptive controller
- Reference Model: $\dot{x}_m = -4\,x_m + 4\,r(t), \quad x_m(0) = 0$
- Adaptive Control: $u = \hat{k}_x\, x + \hat{k}_r\, r$
- Parameter Adaptation:

$$\dot{\hat{k}}_x = -2\,x\,e, \quad \hat{k}_x(0) = 0$$
$$\dot{\hat{k}}_r = -2\,r\,e, \quad \hat{k}_r(0) = 0$$

- Two Reference Inputs:

$$r(t) = 4$$
$$r(t) = 4\sin(3t)$$

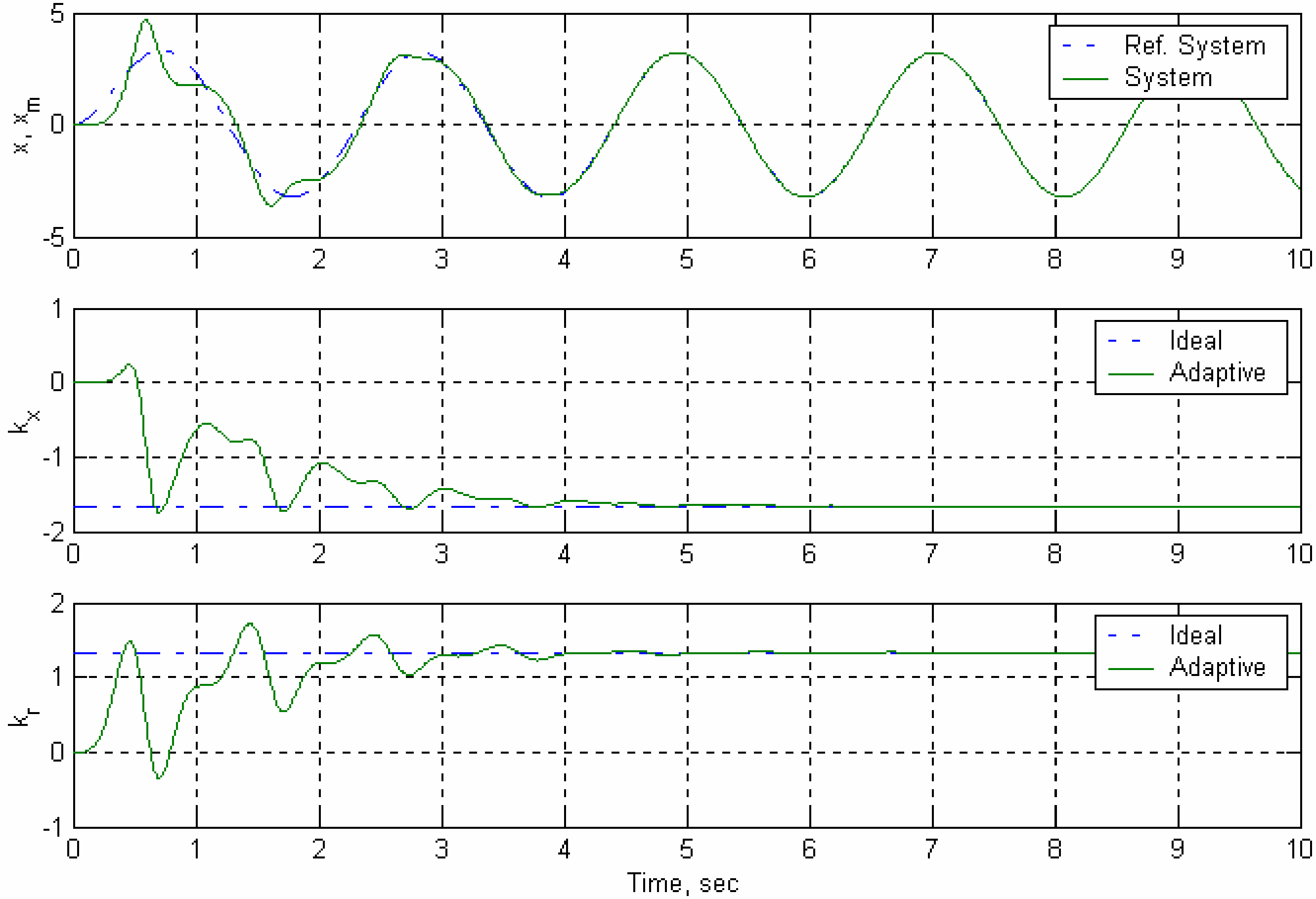
1st-Order *Linear* System
MRAC Simulation w/o PE: $r(t) = 4$



Tracking Error Converges to Zero

Parameter Errors don't Converge to Zero

1st-Order *Linear* System
MRAC Simulation with PE: $r(t) = 4\sin(3t)$



Tracking and Parameter Errors Converge to Zero

Example: MRAC of a 1st-Order *Nonlinear* System

- Unstable Dynamics: $\dot{x} = x + 3\left(u - f(x)\right), \quad x(0) = 0$
  - plant parameters $a = 1, \quad b = 3$ are *unknown*
  - nonlinearity: $f(x) = \theta^T \Phi(x)$
    - *known* basis functions: $\Phi(x) = \begin{pmatrix} x^3 & e^{-(x+0.5)^2 10} & e^{-(x-0.5)^2 10} & \sin(2x) \end{pmatrix}^T$
    - *unknown* parameters: $\theta = \begin{pmatrix} 0.01 & -1 & 1 & 0.5 \end{pmatrix}^T$
- Reference Model: $\dot{x}_m = -4x_m + 4r(t), \quad x_m(0) = 0$
- Adaptive Control: $u = \hat{k}_x x + \hat{k}_r r + \hat{\theta}^T \Phi(x)$
- Parameter Adaptation:

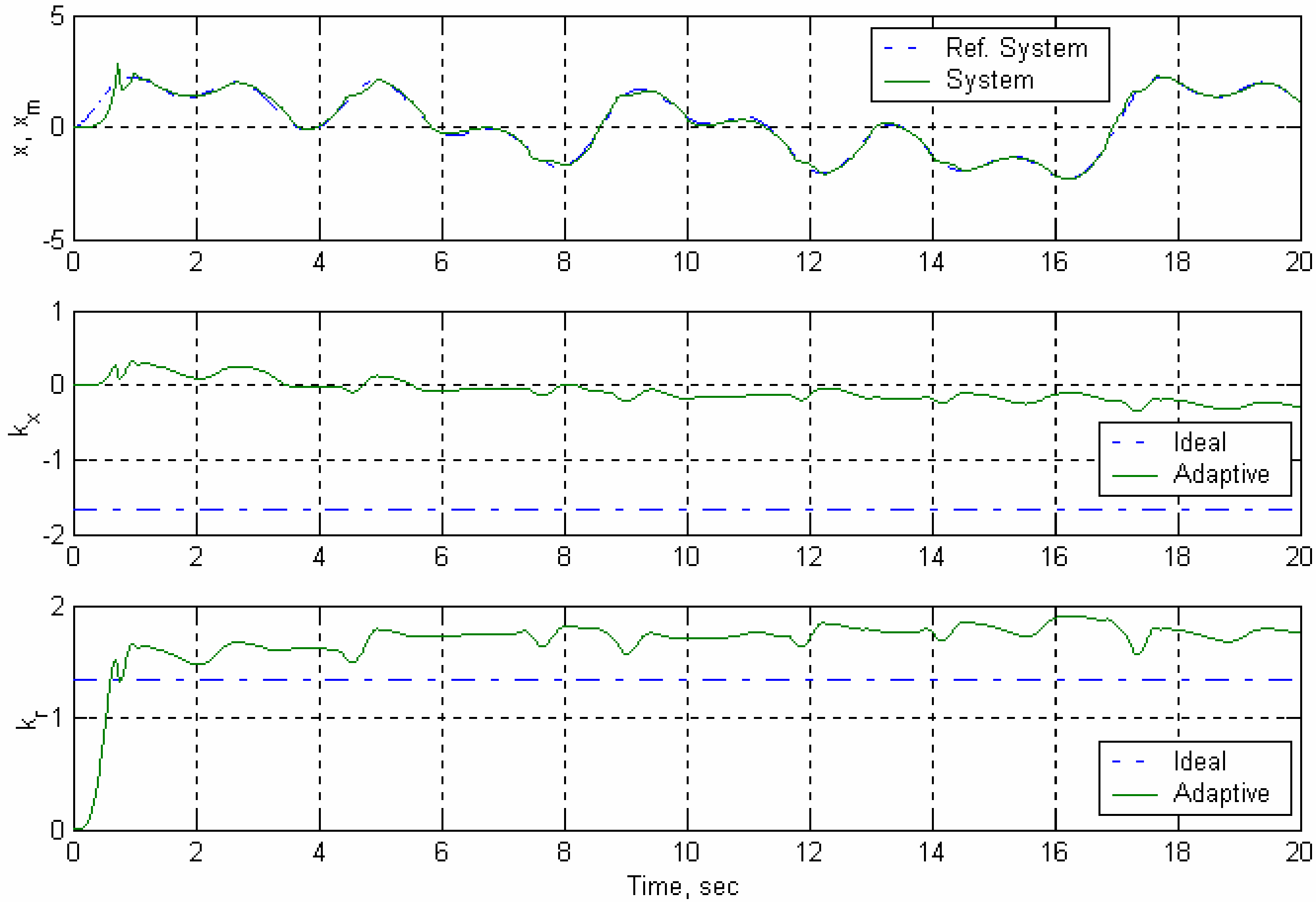
$$\dot{\hat{k}}_x = -2xe, \quad \hat{k}_x(0) = 0$$

$$\dot{\hat{k}}_r = -2re, \quad \hat{k}_r(0) = 0$$

$$\dot{\hat{\theta}} = -2\Phi(x)e, \quad \hat{\theta}(0) = 0_4$$

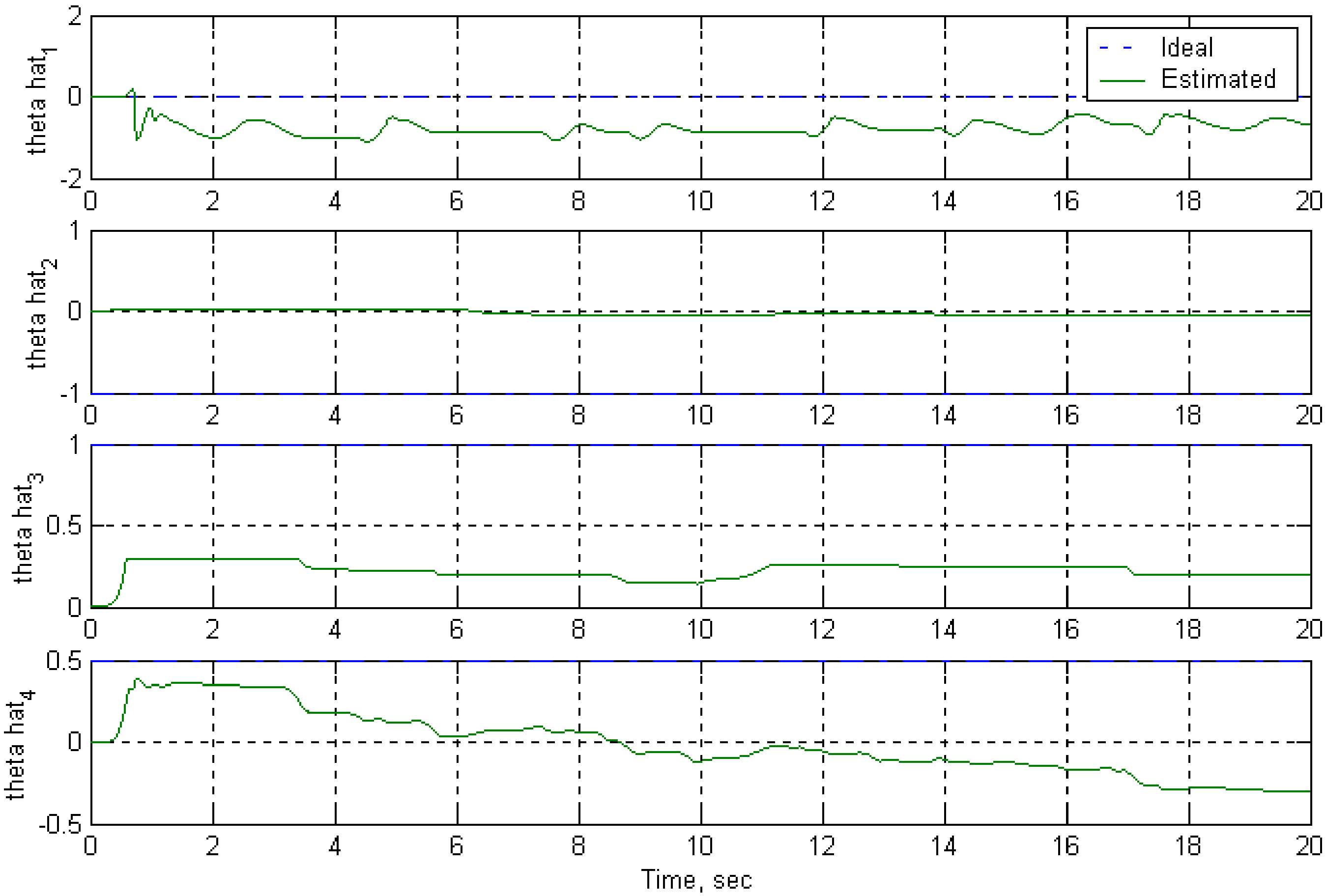
- Reference Input: $r(t) = \sin(3t) + \sin\left(\frac{3t}{2}\right) + \sin\left(\frac{3t}{4}\right) + \sin\left(\frac{3t}{8}\right)$

1st-Order *Nonlinear* System
MRAC Simulation



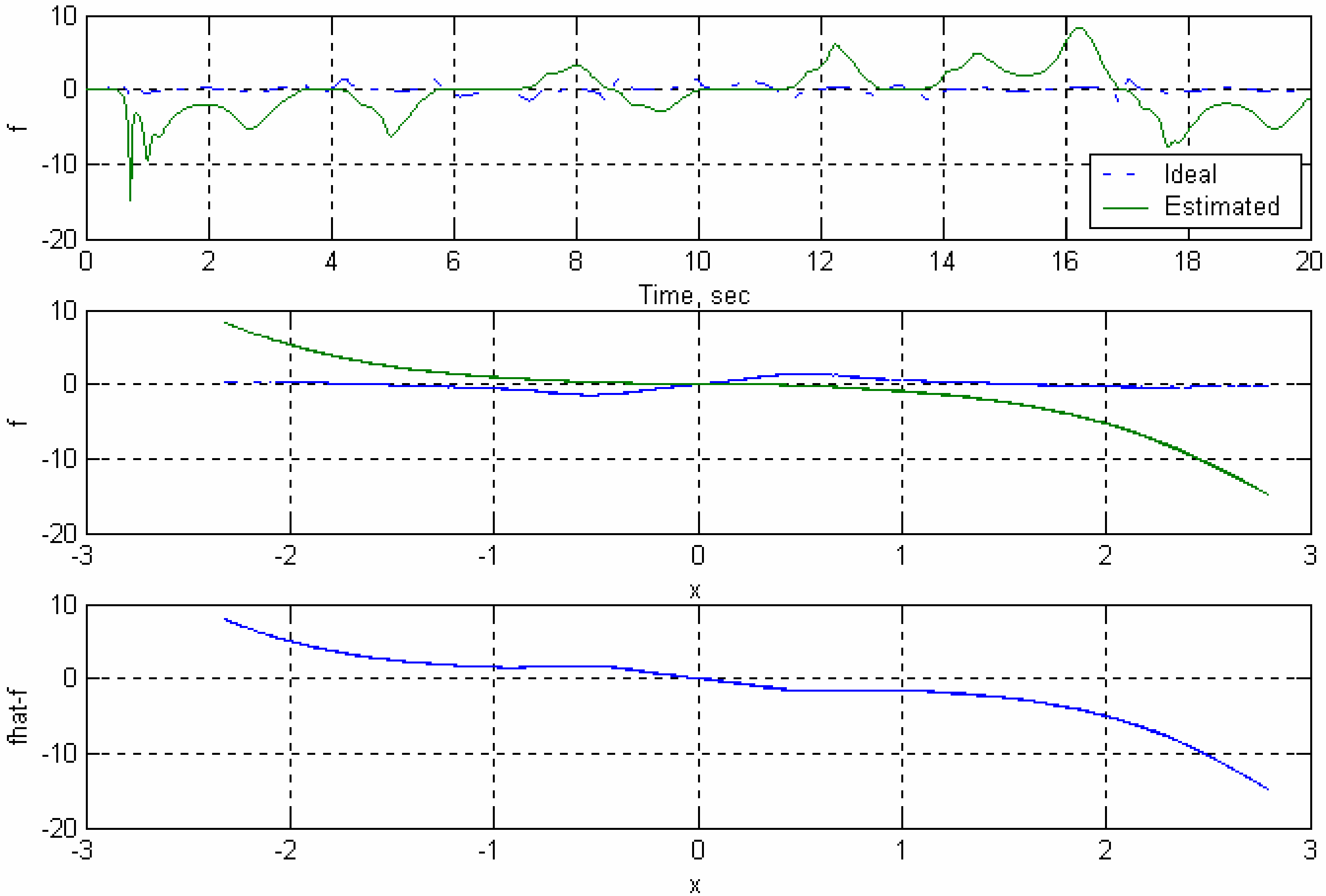
Good Tracking & Poor Parameter Estimation

1^{st}-Order *Nonlinear* System
MRAC Simulation, (continued)



Nonlinearity: Poor Parameter Estimation

1^{st}-Order *Nonlinear* System MRAC Simulation, (completed)



Nonlinearity: Poor Estimation

Example: MRAC of a 1st-Order *Nonlinear* System with *Local* Nonlinearity

- Unstable Dynamics: $\dot{x} = x + 3\left(u - f(x)\right), \quad x(0) = 0$
  - plant parameters $a = 1, \quad b = 3$ are *unknown*
  - nonlinearity: $f(x) = \theta^T \Phi(x)$
    - *known* basis functions: $\Phi(x) = \begin{pmatrix} x^3 & e^{-(x+0.5)^2 10} & e^{-(x-0.5)^2 10} & \sin(2x) \end{pmatrix}^T$
    - *unknown* parameters: → $\theta = \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix}^T$
- Reference Model: $\dot{x}_m = -4x_m + 4r(t), \quad x_m(0) = 0$
- Adaptive Control: $u = \hat{k}_x x + \hat{k}_r r + \hat{\theta}^T \Phi(x)$
- Parameter Adaptation: →

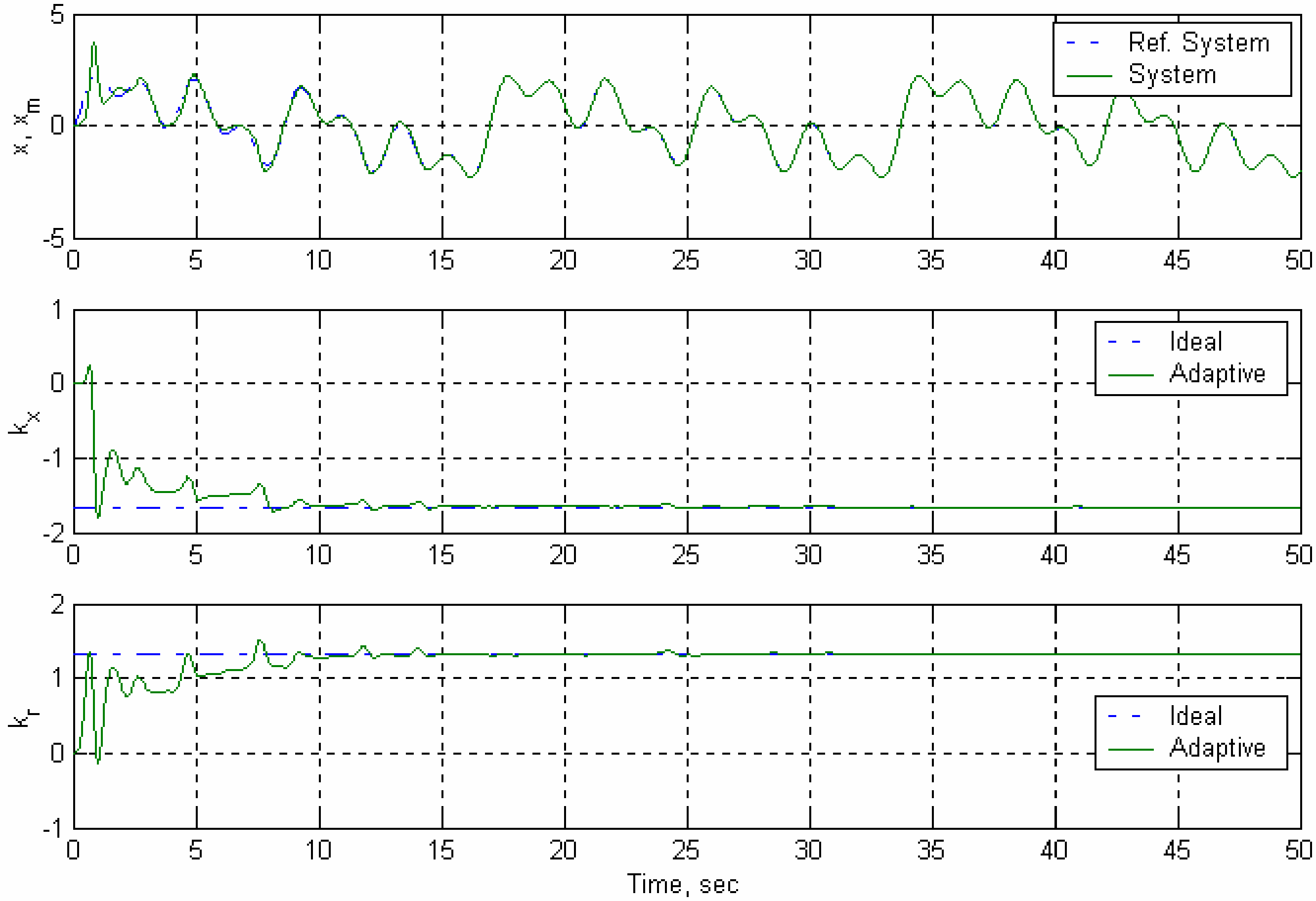
$$\dot{\hat{k}}_x = -2xe, \quad \hat{k}_x(0) = 0$$

$$\dot{\hat{k}}_r = -2re, \quad \hat{k}_r(0) = 0$$

$$\dot{\hat{\theta}} = -2\Phi(x)e, \quad \hat{\theta}(0) = 0_4$$

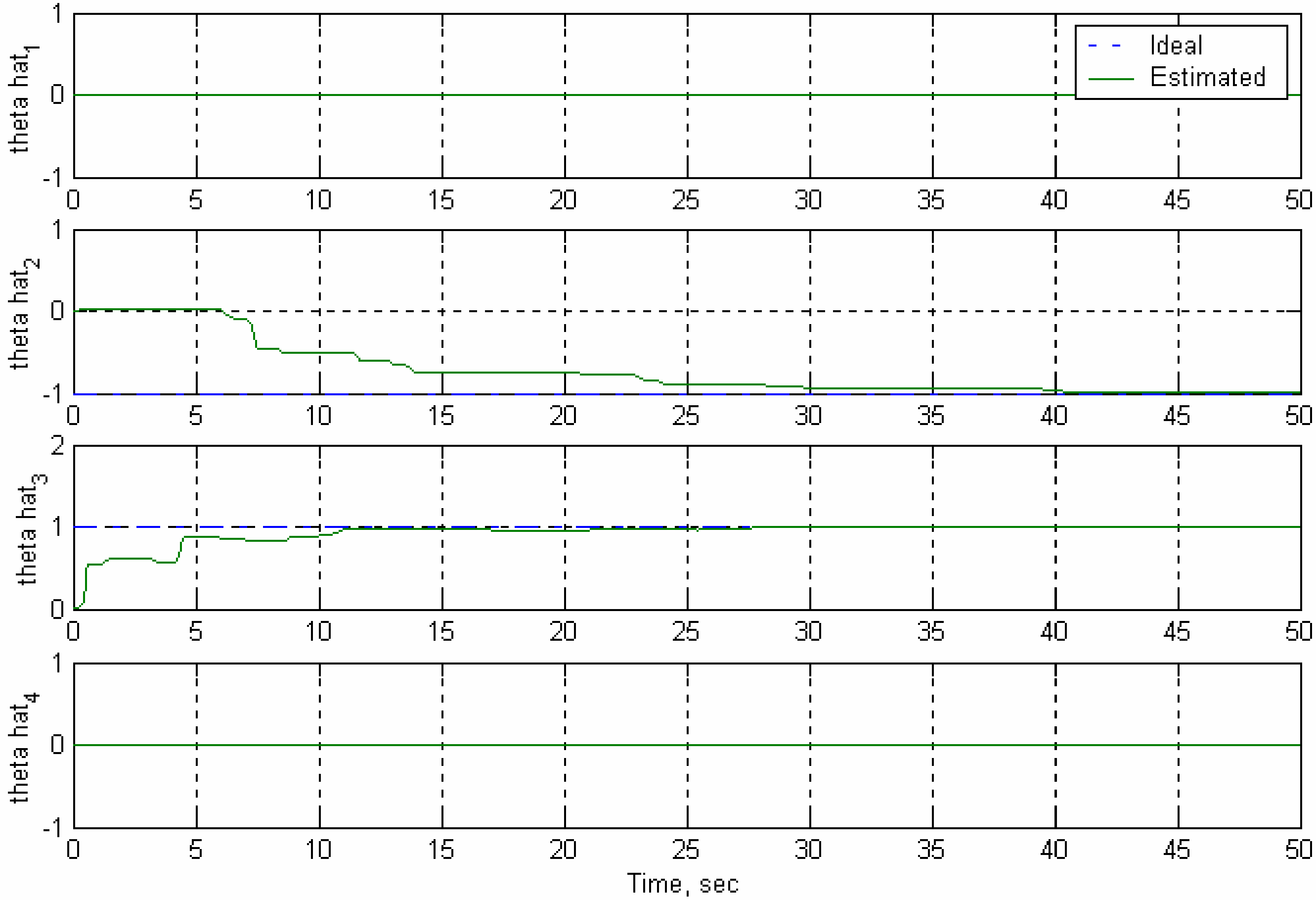
- Reference Input: $r(t) = \sin(3t) + \sin\left(\frac{3t}{2}\right) + \sin\left(\frac{3t}{4}\right) + \sin\left(\frac{3t}{8}\right)$

1^{st}-Order *Nonlinear* System with *Local* Nonlinearity: MRAC Simulation



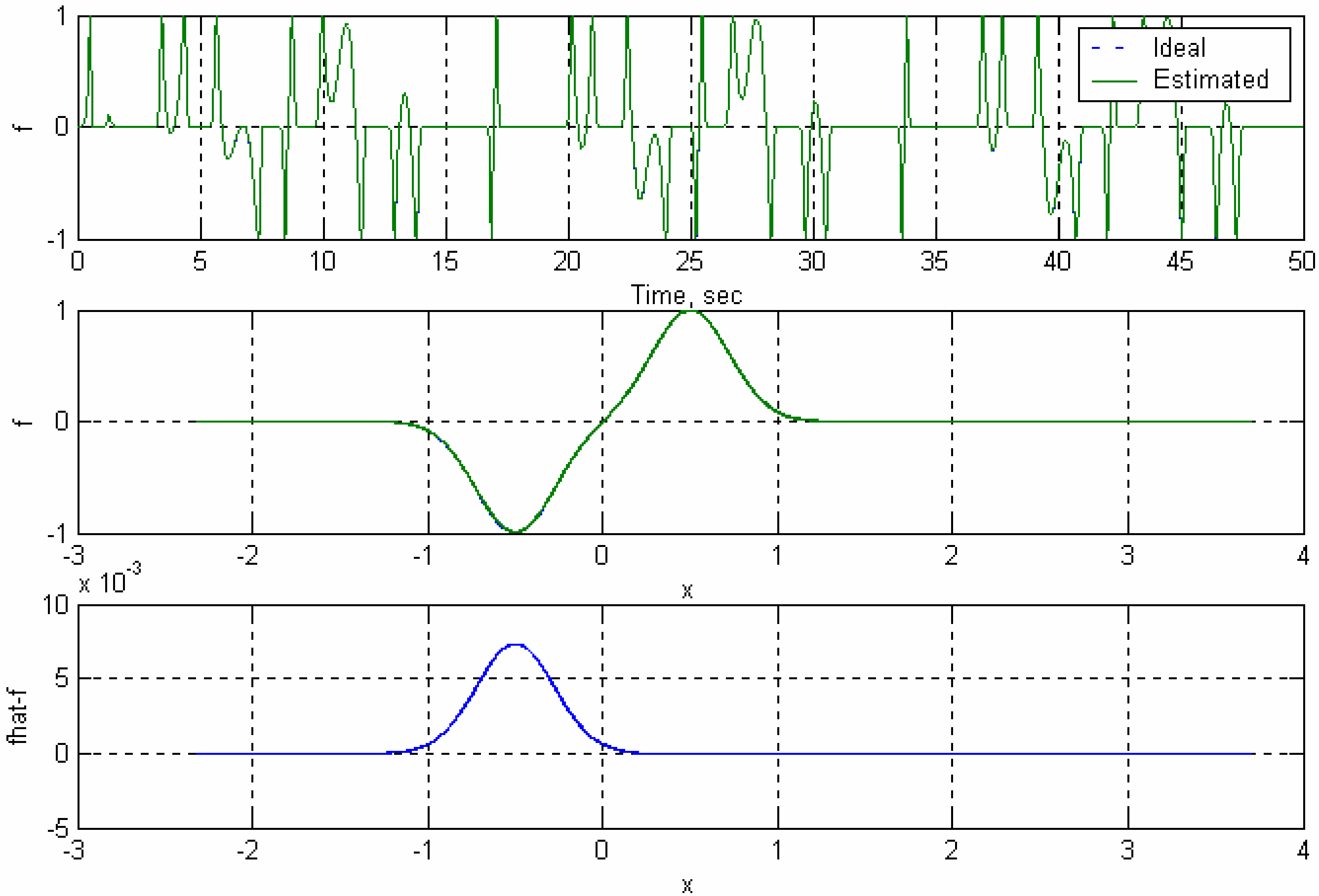
Good Tracking & Parameter Estimation

1st-Order *Nonlinear* System with *Local* Nonlinearity: MRAC Simulation, (continued)



Nonlinearity: Good Parameter Estimation

1st-Order *Nonlinear* System with *Local* Nonlinearity: MRAC Simulation, (completed)



Nonlinearity: Good Function Approximation

MRAC of a 1st-Order *Nonlinear* System
Conclusions & Observations

- *Direct* MRAC provides good tracking in spite of unknown parameters and nonlinear uncertainties in the system dynamics
- Parameter convergence IS NOT guaranteed
- Sufficient Condition for Parameter Convergence
  - Reference input $r(t)$ satisfies Persistency of Excitation
    - PE is hard to verify / compute
  - Enforced for linear systems with *local* nonlinearities
- A control strategy that depends on parameter convergence, (such as *indirect* MRAC), is unreliable, unless PE condition takes place

MRAC Design of n^{th} Order Systems

- System Dynamics: $\dot{x} = A\,x + B\,\Lambda\left(u - f\left(x\right)\right), \quad x \in R^n, \quad u \in R^m$
  - $A \in R^{n\times n}, \Lambda = \operatorname{diag}\left(\lambda_1 \quad \ldots \quad \lambda_m\right) \in R^{m\times m}$ are constant ***unknown*** matrices
  - $B \in R^{n\times m}$ is ***known*** constant matrix
  - $\forall\, i = 1,\ldots,m \quad \operatorname{sgn}\left(\lambda_i\right)$ is ***known***
  - ***uncertain matched*** nonlinear function: $f\left(x\right) = \Theta^T\,\Phi\left(x\right) \in R^m$
    - ***matrix*** of constant ***unknown*** parameters: $\Theta \in R^{m\times N}$
    - vector of *N* ***known*** basis functions: $\Phi\left(x\right) = \left(\varphi_1\left(x\right) \quad \ldots \quad \varphi_N\left(x\right)\right)^T$
- Stable Reference Model: $\dot{x}_m = A_m\,x_m + B_m\,r, \quad \left(A_m \text{ is Hurwitz}\right)$

  $r \in R^m, \quad A_m \in R^{n\times n}, \quad B_m \in R^{n\times m}$
- **Control Goal**
  - find *u* such that: $\lim\limits_{t\to\infty} \left\| x\left(t\right) - x_m\left(t\right) \right\| = 0$

MRAC Design of n^{th} Order Systems (continued)

- Control Feedback: $u = \hat{K}_x^T x + \hat{K}_r^T r + \hat{\Theta}^T \Phi(x)$
  - ($m\,n + m^2 + m\,N$) - parameters to estimate: $\hat{K}_x, \quad \hat{K}_r, \quad \hat{\Theta}$
- Closed-Loop System: $\dot{x} = \left(A + B\Lambda \hat{K}_x^T\right) x + B\Lambda\left(\hat{K}_r^T r + \left(\hat{\Theta} - \Theta\right)^T \Phi(x)\right)$
- Desired Dynamics: $\dot{x}_m = A_m x_m + B_m r$
- Model Matching Conditions
  - there exist *ideal* gains (K_x, K_r) such that: $\Rightarrow$

$$A + B\Lambda K_x^T = A_m$$
$$B\Lambda K_r^T = B_m$$

  - *Note*: knowledge of the ideal gains is not required

$$A + B\Lambda \hat{K}_x^T - A_m = A + B\Lambda \hat{K}_x^T - A - B\Lambda K_x^T = B\Lambda\left(\hat{K}_x - K_x\right)^T = B\Lambda \Delta K_x^T$$
$$B\Lambda \hat{K}_r^T - B_m = B\Lambda \hat{K}_r^T - B\Lambda K_r^T = B\Lambda\left(\hat{K}_r - K_r\right)^T = B\Lambda \Delta \hat{K}_r^T$$

MRAC Design of n^{th} Order Systems (continued)

- Tracking Error: $e(t) = x(t) - x_m(t)$
- Error Dynamics:

$$
\begin{aligned}
\dot{e}(t) &= \dot{x}(t) - \dot{x}_m(t) = \\
&\left(A + B\Lambda \hat{K}_x^T\right)x + B\Lambda\left(\hat{K}_r^T r + \underbrace{\left(\hat{\Theta} - \Theta\right)}_{\Delta\Theta}^T \Phi(x)\right) - A_m x_m - B_m r \pm A_m x \\
&= A_m\left(x - x_m\right) + \left(A + B\Lambda\hat{K}_x^T - A_m\right)x + B\Lambda\left(\hat{K}_r - K_r\right)^T r + B\Lambda\Delta\Theta^T\Phi(x) \\
&= A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r + \Delta\Theta^T \Phi(x)\right)
\end{aligned}
$$

MRAC Design of n^{th} Order Systems (continued)

- Lyapunov Function Candidate

$$V\left(e, \Delta K_x, \Delta K_r, \Delta\Theta\right) = e^T P e$$
$$+\operatorname{trace}\left(\Delta K_x^T \, \Gamma_x^{-1} \, \Delta K_x \left|\Lambda\right|\right) + \operatorname{trace}\left(\Delta K_r^T \, \Gamma_r^{-1} \, \Delta K_r \left|\Lambda\right|\right) + \operatorname{trace}\left(\Delta\Theta^T \, \Gamma_\Theta^{-1} \, \Delta\Theta \left|\Lambda\right|\right)$$

  – where: $\operatorname{trace}(S) \triangleq \sum s_{ii}$
  – $\left|\Lambda\right| \triangleq \operatorname{diag}\left(\left|\lambda_1\right| \quad \ldots \quad \left|\lambda_m^i\right|\right)$ is diagonal matrix with positive elements
  – $\Gamma_x = \Gamma_x^T > 0, \quad \Gamma_r = \Gamma_r^T > 0, \quad \Gamma_\Theta = \Gamma_\Theta^T > 0$ are symmetric positive definite matrices
  – $P = P^T > 0$ is unique symmetric positive definite solution of the algebraic Lyapunov equation $P A_m + A_m^T P = -Q$
    - $Q = Q^T > 0$ is any symmetric positive definite matrix

MRAC Design of n^{th} Order Systems (continued)

- Adaptive Control Design
  - Choose adaptive laws, (on-line parameter updates) such that the time-derivative of the Lyapunov function decreases along the error dynamics trajectories

$$
\begin{aligned}
\dot{\hat{K}}_x &= -\Gamma_x \, x \, e^T P \, B \operatorname{sgn}(\Lambda) \\
\dot{\hat{K}}_r &= -\Gamma_r \, r \, e^T P \, B \operatorname{sgn}(\Lambda) \\
\dot{\hat{\Theta}} &= -\Gamma_\Theta \, \Phi(x) \, e^T P \, B \operatorname{sgn}(\Lambda)
\end{aligned}
$$

- Time-derivative of the Lyapunov function becomes semi-negative definite!

$$
\dot{V}\left(e(t), \Delta K_x(t), \Delta K_r(t), \Delta\Theta(t)\right) = -e^T(t) \, Q \, e(t) \le 0
$$

MRAC Design of n^{th} Order Systems (completed)

- Using Barbalat's and Lyapunov-like Lemmas: $\lim_{t\to\infty} \dot{V}(x,t) = 0$
- Since $\dot{V} = -e^T(t)\,Q\,e^T(t)$ it follows that: $\lim_{t\to\infty} \|e(t)\| = 0$
- **Conclusions**
  - achieved asymptotic tracking: $x(t) \to x_m(t)$, as $t \to \infty$
  - all signals in the closed-loop system are bounded
- ***Remark***
  - Parameter convergence IS NOT guaranteed

Robustness of Adaptive Control

- Adaptive controllers are designed to control real physical systems
  - non-parametric uncertainties may lead to performance degradation and / or instability
    - low-frequency unmodeled dynamics, (structural vibrations)
    - low-frequency unmodeled dynamics, (Coulomb friction)
    - measurement noise
    - computation round-off errors and sampling delays
- Need to enforce robustness of MRAC

Parameter Drift in MRAC

- When $r(t)$ is *persistently exciting* the system, both simulation and analysis indicate that MRAC systems are robust w.r.t non-parametric uncertainties
- When $r(t)$ IS NOT *persistently exciting* even small uncertainties may lead to severe problems
  - estimated parameters drift slowly as time goes on, and suddenly diverge sharply
  - reference input contains insufficient parameter information
  - adaptation has difficulty distinguishing parameter information from noise

Parameter Drift in MRAC: Summary

- Occurs when signals are not persistently exciting
- Mainly caused by measurement noise and disturbances
- Does not effect tracking accuracy until the instability occurs
- Leads to *sudden* failure

Dead-Zone Modification

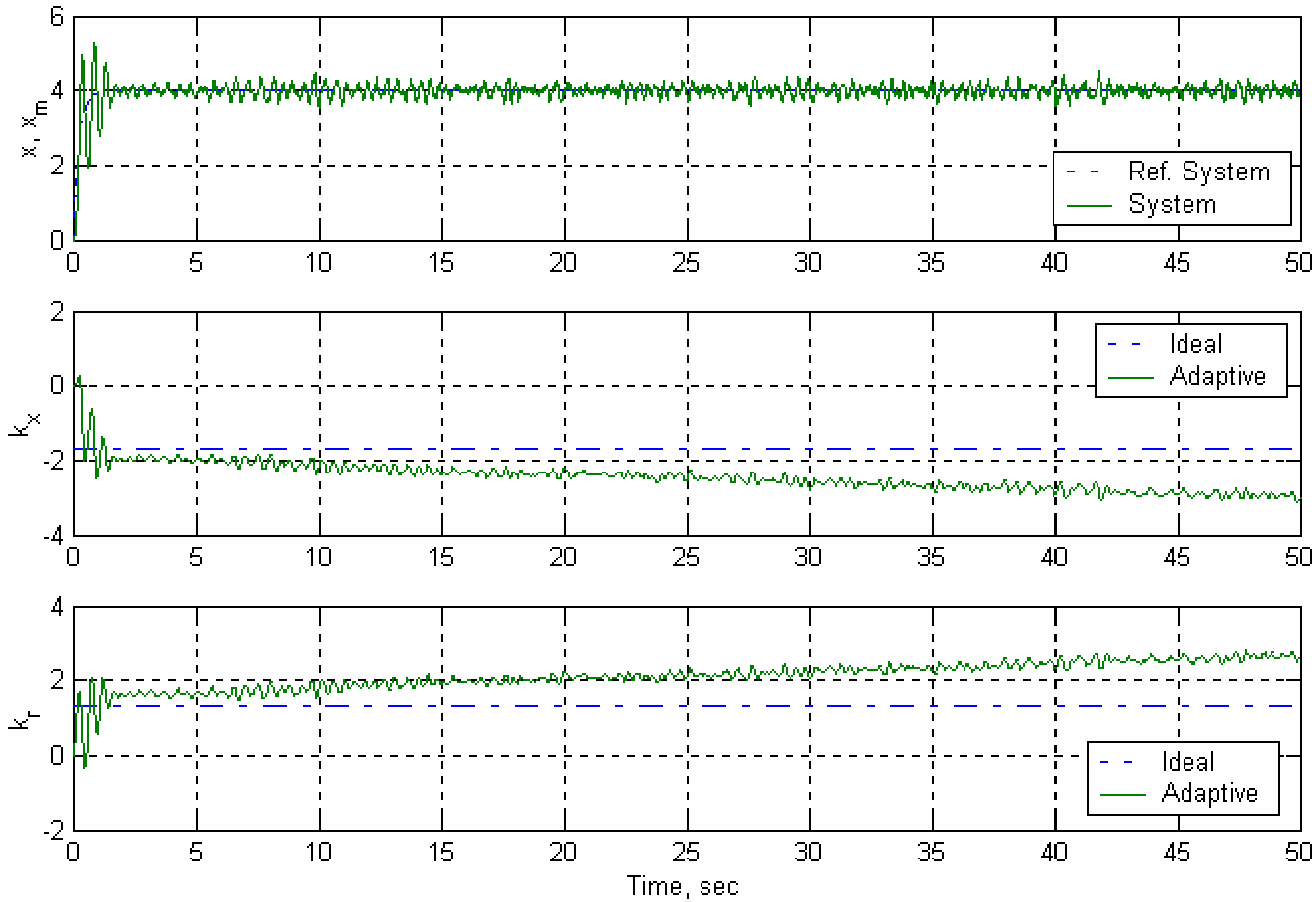
- Method is based on the observation that small tracking errors contain mostly noise and disturbance
- Solution
  - Turn off the adaptation process for "small" tracking errors
  - MRAC using Dead-Zone
  - ε is the size of the dead-zone
- Outcome
  - *Bounded* Tracking

$$\dot{\hat{K}}_x = \begin{cases} -\Gamma_x \, x \, e^T P \, B \, \mathrm{sgn}(\Lambda), & \|e\| > \varepsilon \\ 0, & \|e\| \le \varepsilon \end{cases}$$

$$\dot{\hat{K}}_r = \begin{cases} -\Gamma_r \, r \, e^T P \, B \, \mathrm{sgn}(\Lambda), & \|e\| > \varepsilon \\ 0, & \|e\| \le \varepsilon \end{cases}$$

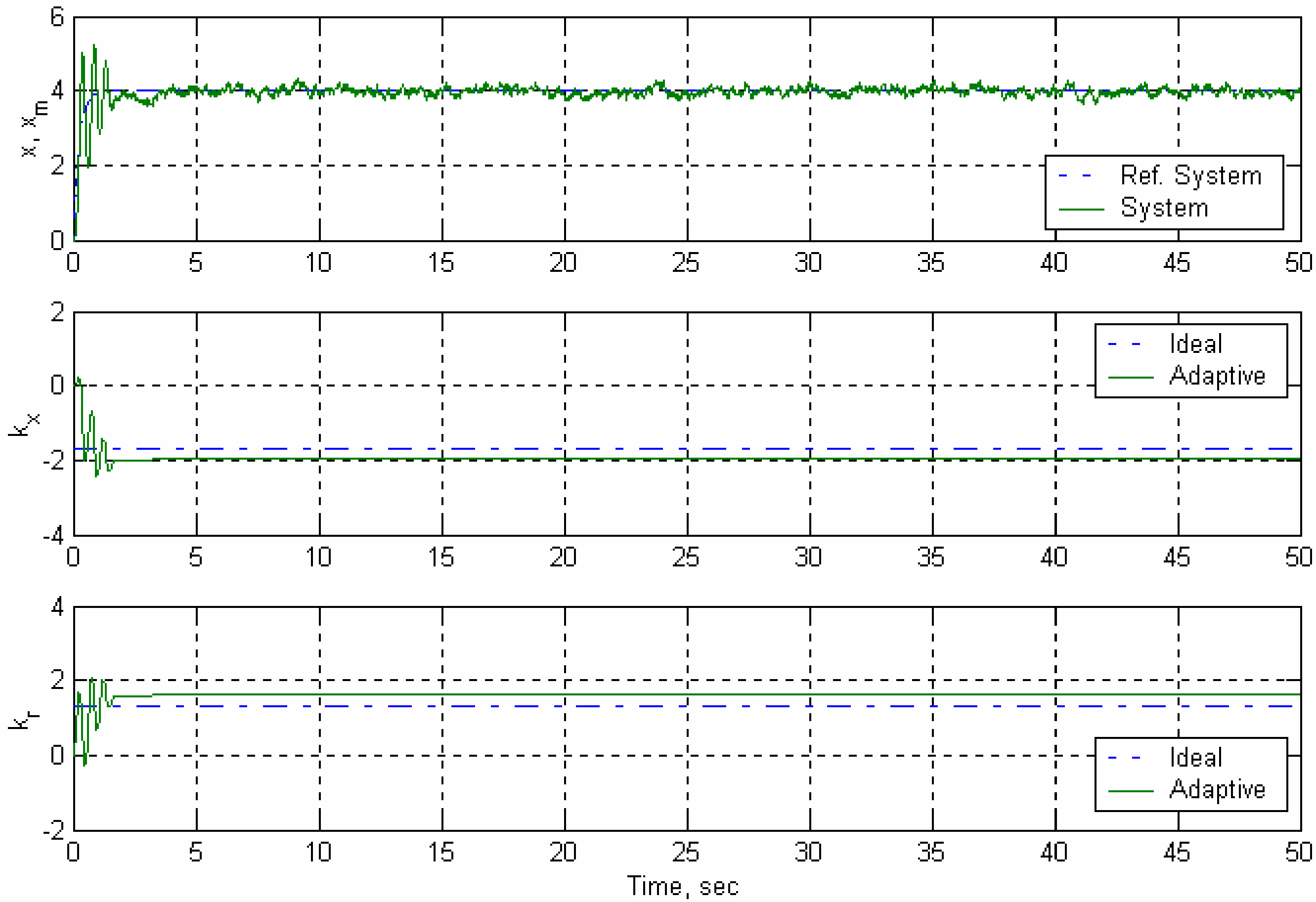
$$\dot{\hat{\Theta}} = \begin{cases} -\Gamma_\Theta \, \Phi(x) \, e^T P \, B \, \mathrm{sgn}(\Lambda), & \|e\| > \varepsilon \\ 0, & \|e\| \le \varepsilon \end{cases}$$

1st-Order *Linear* System with Noise MRAC w/o Dead-Zone: $r(t) = 4$



- Satisfactory Tracking
- **Parameter Drift** due to measurement noise

1st-Order *Linear* System with Noise MRAC *with* Dead-Zone: $r(t) = 4$



- Satisfactory Tracking
- **No Parameter Drift**

Parametric and Non-Parametric Uncertainties

- Parametric Uncertainties are often easy to characterize
  - Example: $m\ddot{x} = u$
    - uncertainty in mass *m* is parametric
    - neglected motor dynamics, measurement noise, sensor dynamics are non-parametric uncertainties
- Both Parametric and Non-Parametric Uncertainties occur during Function Approximation

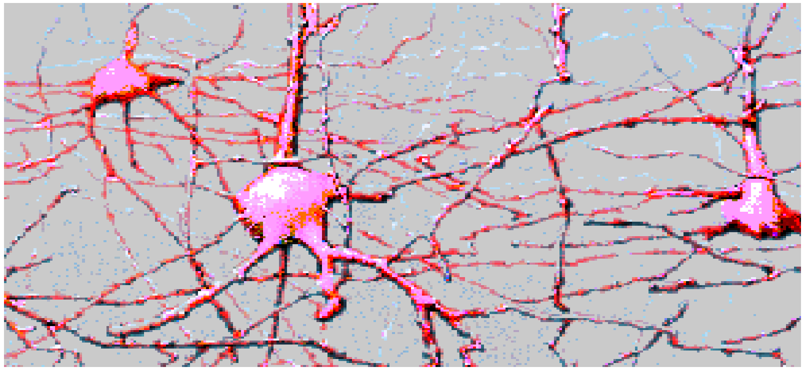
$$\hat{f}(x) = \sum_{i=1}^{N} \theta_i \varphi_i(x) + \varepsilon(x)$$

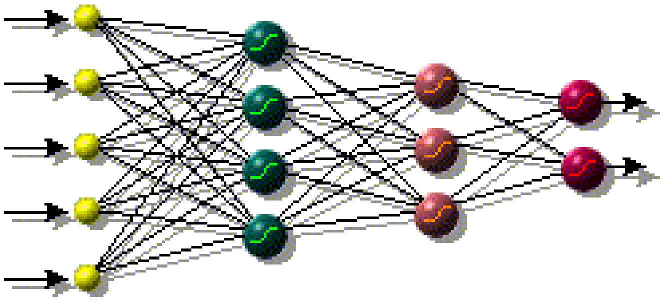
parametric

non-parametric

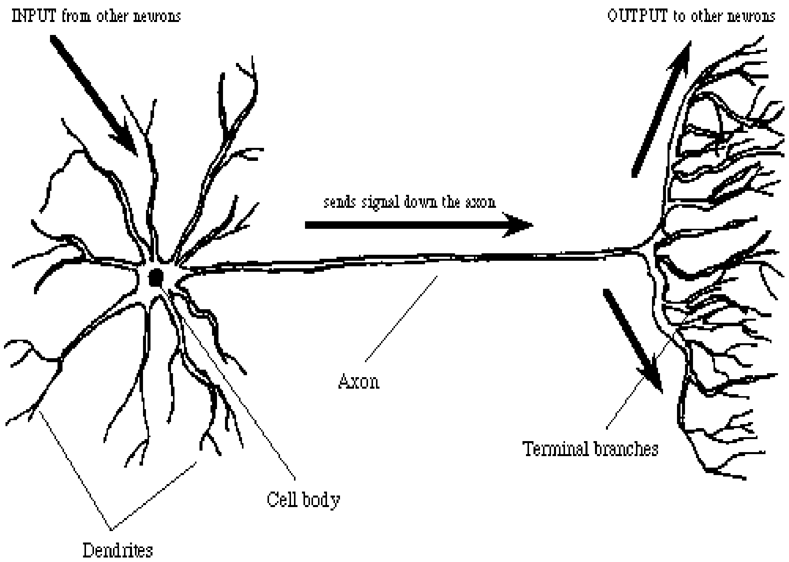
Enforcing Robustness in MRAC Systems

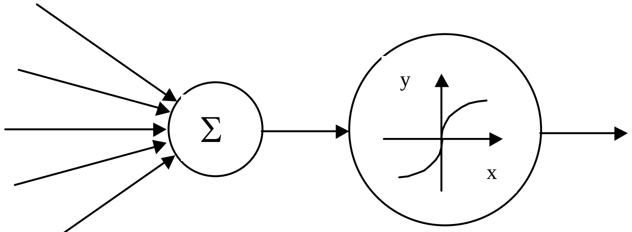
- Non-Parametric Uncertainty
  - Dead-Zone modification
  - Others ?
- Parametric Uncertainty
  - Need a set of basis functions that can approximate a large class of functions within a given tolerance
    - Fourier series
    - Splines
    - Polynomials
    - Artificial Neural Networks
      - sigmoidal
      - RBF





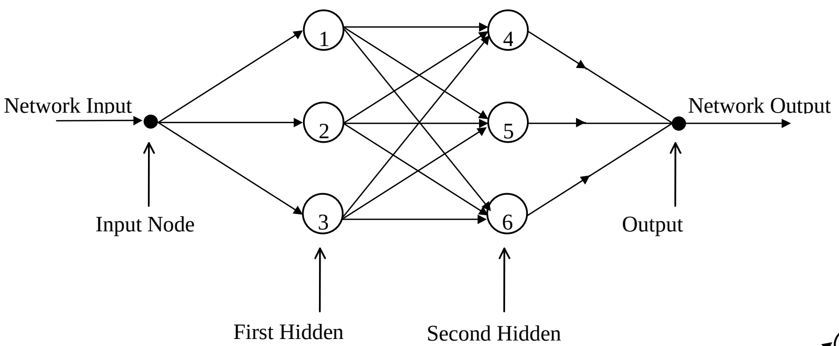
Artificial Neural Networks



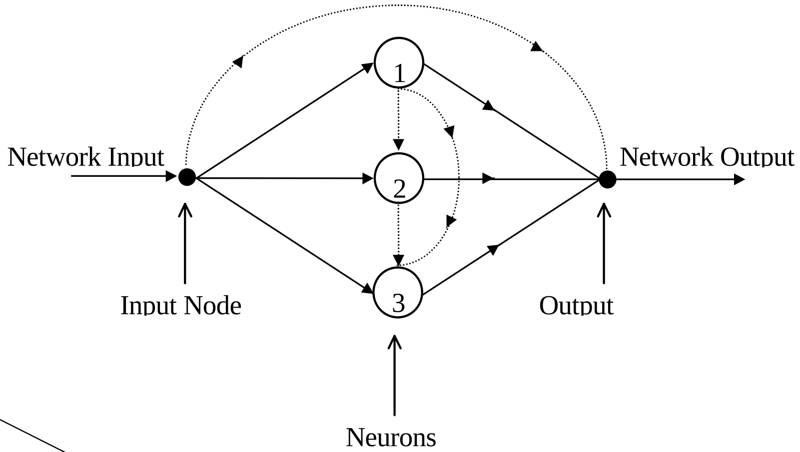


NN Architectures

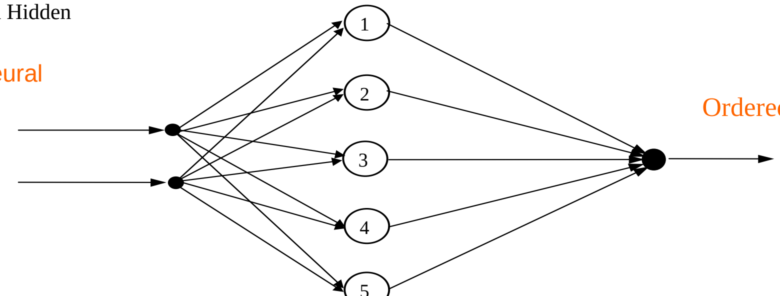
- ***Artificial Neural Networks*** are multi-input-multi-output systems composed of many inter-connected nonlinear processing elements (neurons) operating in parallel



Two-Hidden-Layers Neural Network



Ordered Neural Network



Single-Hidden-Layer Neural Network

Single Hidden Layer (SHL) Feedforward Neural Networks (FNN)

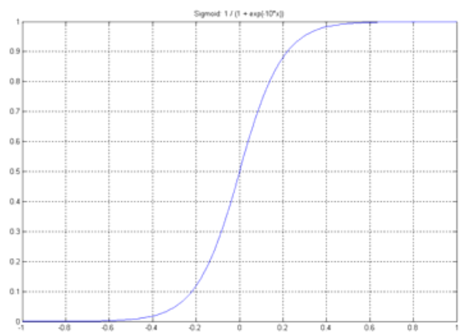
- Three distinct characteristics
  - model of each neuron includes a ***nonlinear*** activation function
    - sigmoid

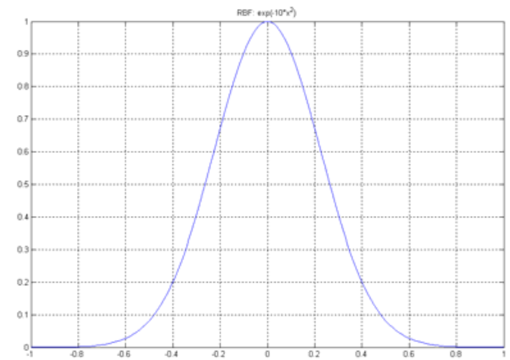
$$\sigma(s) = \frac{1}{1+e^{-s}}$$

    - radial basis function

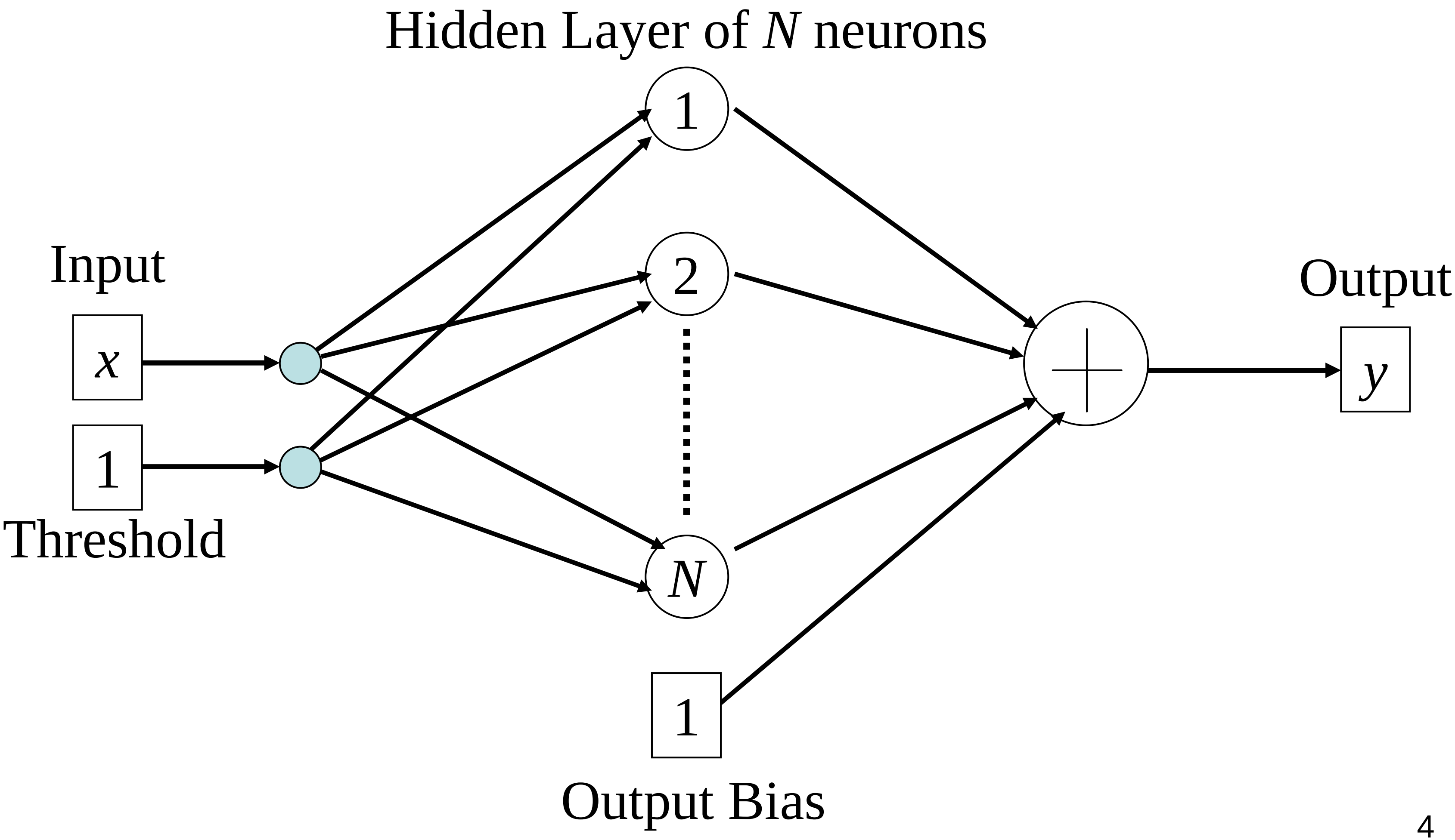
$$\varphi(x) = e^{-\frac{\|x-r\|^2}{2\sigma^2}}$$

  - a single layer of N ***hidden*** neurons
  - ***feedforward*** connectivity





SHL FNN Architecture



SHL FNN Function

- Maps *n* - dimensional input into *m* - dimensional output: $x \to NN(x), \quad x \in R^n, \quad NN(x) \in R^m$
- Functional Dependence
  - sigmoidal: $NN(x) = W^T \vec{\sigma}\left(V^T x + \theta\right) + b$
  - RBF:

$$NN(x) = W^T \underbrace{\begin{pmatrix} \varphi\left(\left\| x - C_1 \right\|\right) \\ \vdots \\ \varphi\left(\left\| x - C_N \right\|\right) \end{pmatrix}}_{\Phi(x)} + b = W^T \Phi(x) + b$$

Sigmoidal NN

- Matrix form: $NN(x) = W^T \vec{\sigma}\left(V^T \begin{pmatrix} x \\ 1 \end{pmatrix}\right) + c$

- Vector of hidden layer sigmoids:

$$\vec{\sigma}\left(V^T x + \theta\right) = \left(\sigma\left(v_1^T x + \theta_1\right) \quad \ldots \quad \sigma\left(\vec{v}_N^T x + \theta_N\right)\right)^T$$

- Matrix of inner-layer weights: $V = \left(\vec{v}_1 \quad \ldots \quad \vec{v}_N\right) \in R^{n \times N}$
- Matrix of output-layer weights: $W = \left(\vec{w}_1 \quad \ldots \quad \vec{w}_m\right) \in R^{N \times m}$
- Vector of output biases $c \in R^m$ and thresholds $\theta \in R^N$
- *k*th output:

$$NN_k(x) = \vec{w}_k^T \sigma\left(\vec{v}_k^T x + \theta_k\right) + c_k = \sum_{j=1}^{N} w_{jk} \sigma\left(\sum_{i=1}^{n} v_{ik} x_i + \theta_k\right) + c_k$$

Sigmoidal NN, (continued)

- Universal Approximation Property
  - large class of functions can be approximated by sigmoidal SHL NN-s within any given tolerance, on compacted domains

$$\forall\, f(x): R^n \to R^m \quad \forall\, \varepsilon > 0 \quad \exists\, N, W, b, V, \theta \quad \forall\, x \in X \subset R^n$$

$$\left\| f(x) - W^T \vec{\sigma}\left(V^T \begin{pmatrix} x \\ 1 \end{pmatrix} \right) - b \right\| \le \varepsilon = \mathrm{O}\left(\frac{1}{\sqrt{N}} \right)$$

- Introduce: $W \triangleq \begin{bmatrix} W^T & b \end{bmatrix}^T, \quad V \triangleq \begin{bmatrix} V^T & \theta \end{bmatrix}^T, \quad \vec{\sigma} \triangleq \begin{bmatrix} \vec{\sigma} \\ 1 \end{bmatrix}, \quad \mu \triangleq \begin{bmatrix} x \\ 1 \end{bmatrix}$
- Then: ⇨ $NN(x) = W^T \vec{\sigma}\left(V^T \mu \right)$

Sigmoidal SHL NN: Summary

- A very large class of functions can be approximated using *linear combinations of shifted and scaled sigmoids*
- NN approximation error decreases as the number of hidden-layer neurons *N* increases:

$$\left\| f(x) - NN(x) \right\| = \mathrm{O}\left(N^{-\frac{1}{2}} \right)$$

- Inclusion of biases and thresholds into NN weight matrices simplifies bookkeeping

$$NN(x) = W^T \vec{\sigma}\left(V^T \mu\right)$$

- Function approximation using sigmoidal NN means finding connection weights *W* and *V*

RBF NN

- Matrix form: $NN(x) = W^T \, \Phi(x) + b$
- Vector of RBF-s: $\Phi(x) = \begin{pmatrix} e^{\frac{-\|x - C_1\|^2}{2\sigma_1^2}} & \ldots & e^{\frac{-\|x - C_N\|^2}{2\sigma_N^2}} \end{pmatrix}^T$
- Matrix of RBF centers: $C \triangleq \begin{bmatrix} \vec{C}_1 & \ldots & \vec{C}_N \end{bmatrix} \in R^{n \times N}$
- Vector of RBF widths: $\vec{\sigma} \triangleq \begin{pmatrix} \sigma_1 & \ldots & \sigma_N \end{pmatrix}^T \in R^N$
- Matrix of output weights: $W = \begin{pmatrix} \vec{w}_1 & \ldots & \vec{w}_m \end{pmatrix} \in R^{N \times m}$
- Vector of output biases: $b \in R^m$
- kth output: $NN_k(x) = \vec{w}_k^T \, \Phi(x) + b_k = \sum_{j=1}^{N} w_{jk} \, e^{\frac{-\|x - C_j\|^2}{2\sigma_j^2}} + b_k$

RBF NN, (continued)

- Universal Approximation Property
  - large class of functions can be approximated by RBF NN-s within any given tolerance, on compacted domains

$$\forall f(x): R^n \to R^m \quad \forall \varepsilon > 0 \quad \exists N, W, \vec{C}, \vec{\sigma} \quad \forall x \in X \subset R^n$$

$$\left\| f(x) - W^T \Phi(x) - b \right\| \le \varepsilon = \mathrm{O}\left(N^{-\frac{1}{n}} \right)$$

- Introduce: $W \triangleq [W \quad b], \quad \Phi(x) \triangleq \begin{bmatrix} \Phi(x) \\ 1 \end{bmatrix}$
- Then: $\Rightarrow$ $NN(x) = W^T \Phi(x)$

RBF NN: Summary

- A very large class of functions can be approximated using *linear combinations of shifted and scaled gaussians*
- NN approximation error decreases as the number of hidden-layer neurons *N* increases:

$$\left\| f(x) - NN(x) \right\| = \mathrm{O}\left(N^{-\frac{1}{n}} \right)$$

- Inclusion of biases into NN output weight matrix simplifies bookkeeping

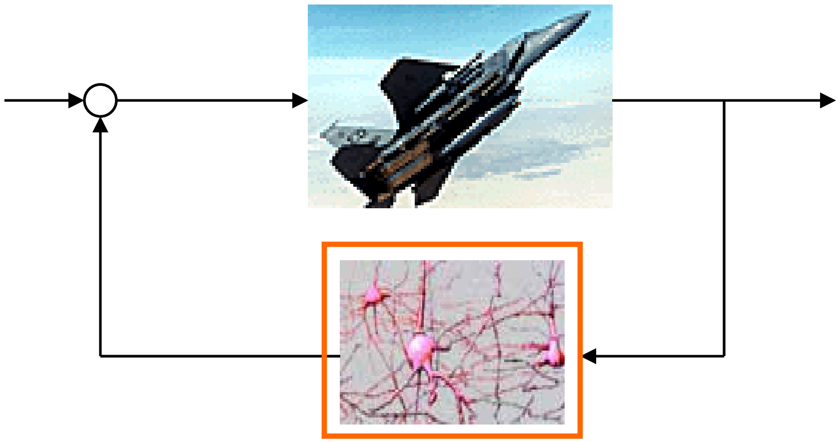
$$NN(x) = W^T \Phi(x)$$

- Function approximation using RBF NN means finding output weights *W*, centers *C*, and widths $\vec{\sigma}$

What is Next?

- Use SHL FNN-s in the context of MRAC systems
  - off-line / on-line approximation of uncertain nonlinearities in system dynamics
    - modeling errors, (aerodynamics)
    - battle damage
    - control failures
- Start with fixed widths RBF NN architectures, (linear in unknown parameters)
- Generalize to using sigmoidal NN-s

Adaptive NeuroControl



n^{th} Order Systems with Matched Uncertainties

- System Dynamics: $$\dot{x} = A x + B \Lambda \left(u - f(x)\right), \quad x \in R^n, \quad u \in R^m$$
  - $A \in R^{n \times n}, \Lambda = \operatorname{diag}\left(\lambda_1 \quad \ldots \quad \lambda_m\right) \in R^{m \times m}$ are constant ***unknown*** matrices
  - $B \in R^{n \times m}$ is ***known*** constant matrix
  - $\forall i = 1, \ldots, m \quad \operatorname{sgn}\left(\lambda_i\right)$ is ***known***
- ***Approximation of uncertainty***: $$f(x) = \Theta^T \Phi(x) + \varepsilon_f(x)$$
  - ***matrix*** of constant ***unknown*** parameters: $\Theta \in R^{m \times N}$
  - vector of *N* ***fixed*** RBF-s: $\Phi(x) = \left(\varphi_1(x) \quad \ldots \quad \varphi_N(x)\right)^T$
  - function approximation tolerance: $\varepsilon_f(x) \in R^m$

n^{th} Order Systems with Matched Uncertainties, (continued)

- *Assumption*: Number of RBF-s, true (unknown) output weights *W* and widths $\vec{\sigma}$ are such that RBF NN approximates the nonlinearity within given tolerance:

$$\left\|\varepsilon_f(x)\right\| = \left\| f(x) - \Theta^T \Phi(x) \right\| \le \varepsilon, \quad \forall x \in X \subset R^n$$

- RBF NN estimator: $\hat{f}(x) = \hat{\Theta}^T \Phi(x)$
- Estimation error:

$$NN(x) - f(x) = \underbrace{\left(\hat{\Theta} - \Theta\right)}_{\Delta\Theta}{}^T \Phi(x) - \varepsilon_f(x) = \Delta\Theta^T \Phi(x) - \varepsilon_f(x)$$

n^{th} Order Systems with Matched Uncertainties, (continued)

- Stable Reference Model: $\dot{x}_m = A_m\, x_m + B_m\, r, \quad (A_m \text{ is Hurwitz})$

  $r \in R^m, \quad A_m \in R^{n\times n}, \quad B_m \in R^{m\times m}$
- **Control Goal**
  - bounded tracking: $\lim\limits_{t\to\infty} \left\| x(t) - x_m(t) \right\| \le \varepsilon_x$
- MRAC Design Process
  - choose *N* and vector of widths $\vec{\sigma}$
    - can be performed off-line in order to incorporate any prior knowledge about the uncertainty
  - design MRAC and evaluate closed-loop system performance
  - repeat previous two steps, if required

n^{th} Order Systems with Matched Uncertainties, (continued)

- Control Feedback: $u = \hat{K}_x^T x + \hat{K}_r^T r + \hat{\Theta}^T \Phi(x)$
  - ($m\,n + m^2 + m\,N$) - parameters to estimate: $\hat{K}_x, \quad \hat{K}_r, \quad \hat{\Theta}$
- Closed-Loop: $\dot{x} = \left(A + B\Lambda \hat{K}_x^T\right)x + B\Lambda\left(\hat{K}_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right)$
- Desired Dynamics: $\dot{x}_m = A_m x_m + B_m r$
- Model Matching Conditions
  - there exist *ideal* gains (K_x, K_r) such that: $\Rightarrow$ $A + B\Lambda K_x^T = A_m$, $B\Lambda K_r^T = B_m$
  - *Note*: knowledge of the ideal gains is not required

$$A + B\Lambda \hat{K}_x^T - A_m = A + B\Lambda \hat{K}_x^T - A - B\Lambda K_x^T = B\Lambda\left(\hat{K}_x - K_x\right)^T = B\Lambda \Delta K_x^T$$

$$B\Lambda \hat{K}_r^T - B_m = B\Lambda \hat{K}_r^T - B\Lambda K_r^T = B\Lambda\left(\hat{K}_r - K_r\right)^T = B\Lambda \Delta \hat{K}_r^T$$

n^{th} Order Systems with Matched Uncertainties, (continued)

- Tracking Error: $e(t) = x(t) - x_m(t)$
- Error Dynamics:

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) =$$
$$\left(A + B\Lambda \hat{K}_x^T\right)x + B\Lambda\left(\hat{K}_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right) - A_m x_m - B_m r \pm A_m x$$
$$= A_m(x - x_m) + \left(A + B\Lambda \hat{K}_x^T - A_m\right)x + B\Lambda\left(\hat{K}_r - K_r\right)^T r + B\Lambda\left(\Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right)$$
$$= A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right)$$

- Remarks
  – estimation error $\varepsilon_f(x)$ is bounded, as long as $x \in X$
  – need to keep x within X

n^{th} Order Systems with Matched Uncertainties, (continued)

- Lyapunov Function Candidate

$$V\left(e, \Delta K_x, \Delta K_r, \Delta\Theta\right) = e^T P e + \operatorname{trace}\left(\Delta K_x^T \Gamma_x^{-1} \Delta K_x |\Lambda|\right) + \operatorname{trace}\left(\Delta K_r^T \Gamma_r^{-1} \Delta K_r |\Lambda|\right) + \operatorname{trace}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \Delta\Theta |\Lambda|\right)$$

  - where: $\operatorname{trace}(S) \triangleq \sum s_{ii}$
  - $|\Lambda| \triangleq \operatorname{diag}\left(|\lambda_1| \quad \ldots \quad |\lambda_m^i|\right)$ is diagonal matrix with positive elements
  - $\Gamma_x = \Gamma_x^T > 0, \quad \Gamma_r = \Gamma_r^T > 0, \quad \Gamma_\Theta = \Gamma_\Theta^T > 0$ are symmetric positive definite matrices
  - $P = P^T > 0$ is unique symmetric positive definite solution of the algebraic Lyapunov equation $\boxed{P A + A^T P = -Q}$
    - $Q = Q^T > 0$ is any symmetric positive definite matrix

n^{th} Order Systems with Matched Uncertainties, (continued)

- Time-derivative of the Lyapunov function

$$
\begin{aligned}
\dot{V} &= \dot{e}^T P e + e^T P \dot{e} \\
&+2\,\mathrm{trace}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} |\Lambda|\right) \\
&= \left(A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right)\right)^T P e \\
&+e^T P\left(A_m e + B\Lambda\left(\Delta K_x^T x + \Delta K_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right)\right) \\
&+2\,\mathrm{trace}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} |\Lambda|\right) \\
&= e^T \left(A_m P + P A_m\right) e \\
&+2 e^T P B \Lambda\left(\Delta K_x^T x + \Delta K_r^T r + \Delta\Theta^T \Phi(x) - \varepsilon_f(x)\right) \\
&+2\,\mathrm{trace}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r |\Lambda|\right) + 2\,\mathrm{trace}\left(\Delta\Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} |\Lambda|\right)
\end{aligned}
$$

n^{th} Order Systems with Matched Uncertainties, (continued)

- Time-derivative of the Lyapunov function

$$
\begin{aligned}
\dot{V} &= -e^T Q e - 2 e^T P B \Lambda \varepsilon_f(x) \\
&+ 2 e^T P B \Lambda \Delta K_x^T x + 2\,\mathrm{trace}\left(\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x |\Lambda|\right) \\
&+ 2 e^T P B \Lambda \Delta K_r^T r + 2\,\mathrm{trace}\left(\Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r |\Lambda|\right) \\
&+ 2 e^T P B \Lambda \Delta \Theta^T \Phi(x) + 2\,\mathrm{trace}\left(\Delta \Theta^T \Gamma_\Theta^{-1} \dot{\hat{\Theta}} |\Lambda|\right)
\end{aligned}
$$

- Using trace identity: $a^T b = \mathrm{trace}\left(b\, a^T\right)$

- Example: $\underbrace{e^T P B \Lambda}_{a^T} \underbrace{\Delta K_x^T x}_{b} = \mathrm{trace}\left(\underbrace{\Delta K_x^T x}_{b} \underbrace{e^T P B \Lambda}_{a^T}\right)$

n^{th} Order Systems with Matched Uncertainties, (continued)

- Time-derivative of the Lyapunov function

$$
\begin{aligned}
\dot{V} &= -e^T Q e - 2 e^T P B \Lambda \varepsilon_f(x) \\
&+ 2\,\mathrm{trace}\left(\Delta K_x^T \left\{ \Gamma_x^{-1} \dot{\hat{K}}_x + x e^T P B \,\mathrm{sgn}(\Lambda) \right\} |\Lambda| \right) \\
&+ 2\,\mathrm{trace}\left(\Delta K_r^T \left\{ \Gamma_r^{-1} \dot{\hat{K}}_r + r e^T P B \,\mathrm{sgn}(\Lambda) \right\} |\Lambda| \right) \\
&+ 2\,\mathrm{trace}\left(\Delta \Theta^T \left\{ \Gamma_\Theta^{-1} \dot{\hat{\Theta}} + \Phi(x) e^T P B \,\mathrm{sgn}(\Lambda) \right\} |\Lambda| \right)
\end{aligned}
$$

- Problem
  - choose adaptive parameters $\hat{K}_x, \hat{K}_r, \hat{\Theta}$ such that time-derivative $\dot{V}$ becomes negative definite outside of a compact set in the *error* state space, and all parameters remain bounded for all future times

n^{th} Order Systems with Matched Uncertainties, (continued)

- Suppose that we choose adaptive laws:

$$\begin{aligned}\dot{\hat{K}}_x &= -\Gamma_x\, x\, e^T P\, B\,\mathrm{sgn}(\Lambda)\\ \dot{\hat{K}}_r &= -\Gamma_r\, r\, e^T P\, B\,\mathrm{sgn}(\Lambda)\\ \dot{\hat{\Theta}} &= -\Gamma_\Theta\, \Phi(x)\, e^T P\, B\,\mathrm{sgn}(\Lambda)\end{aligned}$$

- Then we get:

$$\dot{V} = -e^T Q\, e - 2\, e^T P\, B\, \Lambda\, \varepsilon_f(x) \le -\lambda_{\min}(Q)\|e\|^2 + 2\|e\|\|P\, B\|\lambda_{\max}(\Lambda)\,\varepsilon$$

- Consequently, $\dot{V} < 0$ outside of the compact set

$$E \triangleq \left\{ e : \|e\| \le \frac{2\|P\, B\|\lambda_{\max}(\Lambda)\,\varepsilon}{\lambda_{\min}(Q)} \right\}$$

- ***Unfortunately***, inside *E* parameter errors may grow out of bounds, (for $e \in E$, $\dot{V}$ IS NOT necessarily negative!)

How to Keep Adaptive Parameters Bounded?

- σ - modification:

$$\dot{\hat{K}}_x = -\Gamma_x \left(x e^T P B + \sigma_x \hat{K}_x \right) \mathrm{sgn}(\Lambda)$$
$$\dot{\hat{K}}_r = -\Gamma_r \left(r e^T P B + \sigma_r \hat{K}_r \right) \mathrm{sgn}(\Lambda)$$
$$\dot{\hat{\Theta}} = -\Gamma_\Theta \left(\Phi(x) e^T P B + \sigma_\Theta \hat{\Theta} \right) \mathrm{sgn}(\Lambda)$$

- e - modification:

$$\dot{\hat{K}}_x = -\Gamma_x \left(x e^T P B + \sigma_x \left\| e^T P B \right\| \hat{K}_x \right) \mathrm{sgn}(\Lambda)$$
$$\dot{\hat{K}}_r = -\Gamma_r \left(r e^T P B + \sigma_r \left\| e^T P B \right\| \hat{K}_r \right) \mathrm{sgn}(\Lambda)$$
$$\dot{\hat{\Theta}} = -\Gamma_\Theta \left(\Phi(x) e^T P B + \sigma_\Theta \left\| e^T P B \right\| \hat{\Theta} \right) \mathrm{sgn}(\Lambda)$$

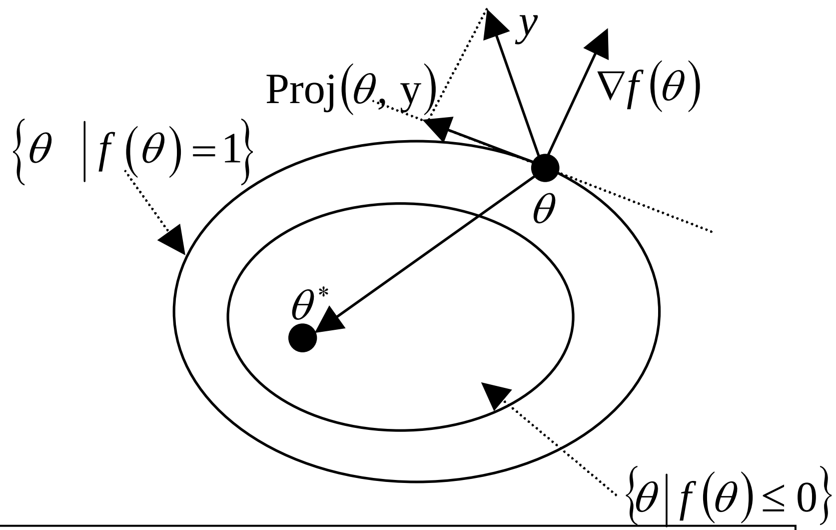
- Modifications add *damping* to adaptive laws
  - damping controlled by choosing $\sigma_x, \sigma_r, \sigma_\Theta > 0$
  - there is a *trade off* between adaptation rate and damping

Introducing Projection Operator

- Requires no damping terms
- Designed to keep NN weights within *pre-specified* bounds
- Maintains negative values of the Lyapunov function time-derivative outside of compact subset: $E \triangleq \left\{ e : \|e\| \le \frac{2\|PB\| \lambda_{\max}(\Lambda)\varepsilon}{\lambda_{\min}(Q)} \right\}$
  – the size of *E* defines tracking tolerance
  – the size of *E* can be controlled!

Projection Operator

- Function $f(\theta)$ defines pre-specified parameter domain boundary
- Example:



$$f(\theta) = \frac{\|\theta\|^2 - \theta_{\max}^2}{\varepsilon_\theta \, \theta_{\max}^2}$$

$$\{f(\theta) \le 0\} \Rightarrow \{\|\theta\| \le \theta_{\max}\} \Rightarrow \theta \text{ is within bounds}$$

$$\{0 < f(\theta) \le 1\} \Rightarrow \left\{\|\theta\| \le \sqrt{1+\varepsilon_\theta}\, \theta_{\max}\right\} \Rightarrow \theta \text{ is within } \left(\sqrt{1+\varepsilon_\theta}\right)\% \text{ of bounds}$$

$$\{f(\theta) > 1\} \Rightarrow \left\{\|\theta\| > \sqrt{1+\varepsilon_\theta}\, \theta_{\max}\right\} \Rightarrow \theta \text{ is outside of bounds}$$

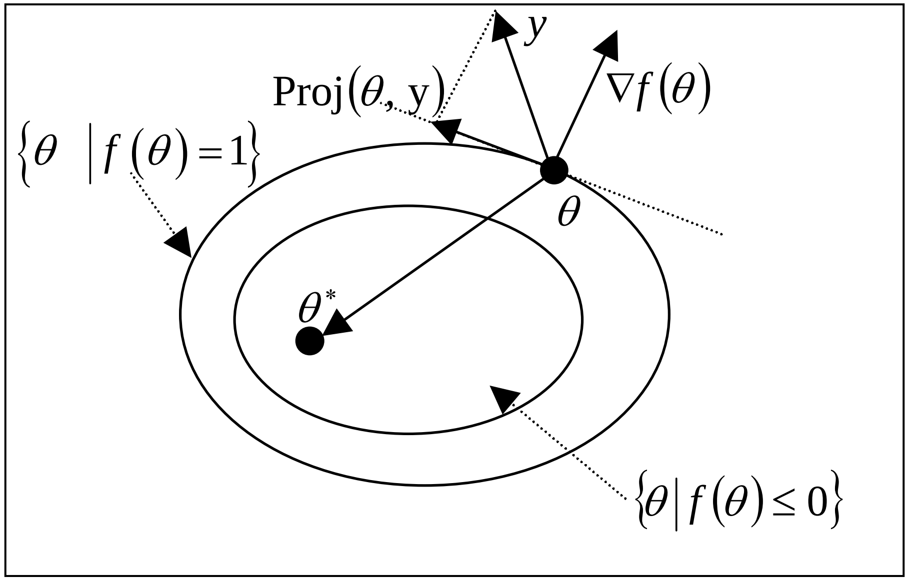
- $\theta_{\max}$ specifies boundary
- ε_θ specifies boundary tolerance

Projection Operator, (continued)

- ***Definition***:

$$\mathrm{Proj}(\theta, y) = \begin{cases} y - \dfrac{\nabla f(\theta)\left(\nabla f(\theta)\right)^T}{\left\|\nabla f(\theta)\right\|^2} y\, f(\theta), & \text{if } f(\theta) > 0 \text{ and } y^T \nabla f(\theta) > 0 \\ y, & \text{if not} \end{cases}$$

- Depends on (θ, y)
- Does not alter *y* if θ is within the pre-specified bounds: $\|\theta\| \le \theta_{\max}$
- Gradient: $\nabla f(\theta) = \dfrac{2\theta}{\varepsilon_\theta \theta_{\max}^2}$
- In $\{0 \le f(\theta) \le 1\}$ the operator subtracts gradient vector $\nabla f(\theta)$ (normal to the boundary) from *y*
  - get a *smooth* transition from *y* for $\lambda = 0$ to a tangent vector field for $\lambda = 1$



- ***Important Property*** $\left(\theta - \theta^*\right)^T \left(\mathrm{Proj}(\theta, y) - y\right) \le 0$

Lyapunov Function Time-Derivative with Projection Operator

- Make trace terms semi-negative *AND* keep parameters bounded:

$$\begin{aligned}
\dot{V} &= -e^T Q e - 2 e^T P B \Lambda \varepsilon_f(x) \\
&+ 2\,\mathrm{trace}\left(\Delta K_x^T \left\{ \underbrace{\Gamma_x^{-1} \dot{\hat{K}}_x}_{\mathrm{Proj}(\hat{K}_x, y)} + \underbrace{x e^T P B \,\mathrm{sgn}(\Lambda)}_{-y} \right\} |\Lambda| \right) \\
&+ 2\,\mathrm{trace}\left(\Delta K_r^T \left\{ \underbrace{\Gamma_r^{-1} \dot{\hat{K}}_r}_{\mathrm{Proj}(\hat{K}_r, y)} + \underbrace{r e^T P B \,\mathrm{sgn}(\Lambda)}_{-y} \right\} |\Lambda| \right) \\
&+ 2\,\mathrm{trace}\left(\Delta \Theta^T \left\{ \underbrace{\Gamma_\Theta^{-1} \dot{\hat{\Theta}}}_{\mathrm{Proj}(\hat{\Theta}, y)} + \underbrace{\Phi(x) e^T P B \,\mathrm{sgn}(\Lambda)}_{-y} \right\} |\Lambda| \right)
\end{aligned}$$

Adaptation with Projection

- Modified adaptive laws:

$$\begin{aligned}\dot{\hat{K}}_x &= \Gamma_x \operatorname{Proj}\left(\hat{K}_x, -x\, e^T P\, B \operatorname{sgn}(\Lambda)\right)\\ \dot{\hat{K}}_r &= \Gamma_r \operatorname{Proj}\left(\hat{K}_r, -r\, e^T P\, B \operatorname{sgn}(\Lambda)\right)\\ \dot{\hat{\Theta}} &= \Gamma_\Theta \operatorname{Proj}\left(\hat{\Theta}, -\Phi(x)\, e^T P\, B \operatorname{sgn}(\Lambda)\right)\end{aligned}$$

- Projection Operator, its bounds and tolerances are defined ***column-wise***
- Lyapunov function time-derivative:

$$\dot{V} \le -e^T Q\, e - 2\, e^T P\, B\, \Lambda\, \varepsilon_f(x) \le -\lambda_{\min}(Q)\|e\|^2 + 2\|e\|\|P\,B\|\lambda_{\max}(\Lambda)\,\varepsilon$$

- Adaptive parameters stay within the pre-specified bounds, while $\dot{V} < 0$ outside of the compact set:

$$E \triangleq \left\{ e : \|e\| \le \frac{2\|P\,B\|\lambda_{\max}(\Lambda)\,\varepsilon}{\lambda_{\min}(Q)} \right\}$$

Example: Projection Operator, (scalar case)

- Scalar adaptive gain: $\dot{\hat{k}} = \gamma \operatorname{Proj}\left(\hat{k}, -x\, e \operatorname{sgn}(b)\right)$
- Pre-specified parameter domain boundary:
  - using function: $f\left(\hat{k}\right) = \dfrac{\hat{k}^2 - k_{\max}^2}{\varepsilon\, k_{\max}^2}$ ⇒ $\nabla f\left(\hat{k}\right) = f'\left(\hat{k}\right) = \dfrac{2\hat{k}}{\varepsilon\, k_{\max}^2}$

$$\left\{ f\left(\hat{k}\right) \le 0 \right\} \Rightarrow \left\{ \left|\hat{k}\right| \le k_{\max} \right\} \Rightarrow \hat{k} \text{ is within bounds}$$

$$\left\{ 0 < f\left(\hat{k}\right) \le 1 \right\} \Rightarrow \left\{ \left|\hat{k}\right| \le \sqrt{1+\varepsilon}\, k_{\max} \right\} \Rightarrow \hat{k} \text{ is within } \left(\sqrt{1+\varepsilon}\right)\% \text{ of bounds}$$

$$\left\{ f\left(\hat{k}\right) > 1 \right\} \Rightarrow \left\{ \left|\hat{k}\right| > \sqrt{1+\varepsilon}\, k_{\max} \right\} \Rightarrow \hat{k} \text{ is outside of bounds}$$

  - Projection Operator:

$y = -x\, e \operatorname{sgn}(b)$ ⇒ $\operatorname{Proj}\left(\hat{k}, y\right) = \begin{cases} y\left(1 - f\left(\hat{k}\right)\right), & \text{if } f\left(\hat{k}\right) > 0 \text{ and } y\, f'\left(\hat{k}\right) > 0 \\ y, & \text{if not} \end{cases}$

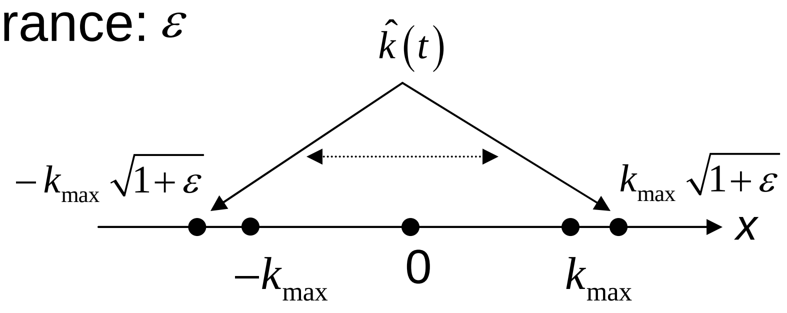
Example: Projection Operator, (scalar case) (continued)

- Adaptive Law, ($b > 0$):

$$\dot{\hat{k}} = \begin{cases} -x\,e\left(1 - f\left(\hat{k}\right)\right), & \text{if} \quad \left[f\left(\hat{k}\right) > 0 \quad \text{and} \quad x\,e\,f'\left(\hat{k}\right) \right] < 0 \\ -x\,e, & \text{if not} \end{cases}$$

$$\text{where:} \quad f\left(\hat{k}\right) = \frac{\hat{k}^2 - k_{\max}^2}{\varepsilon\, k_{\max}^2}$$

- Geometric Interpretation
  - adaptive parameter $\hat{k}(t)$ changes within the pre-specified interval
  - interval bound: $k_{\max}$
  - Bound tolerance: ε



Adaptive *Augmentation* Design

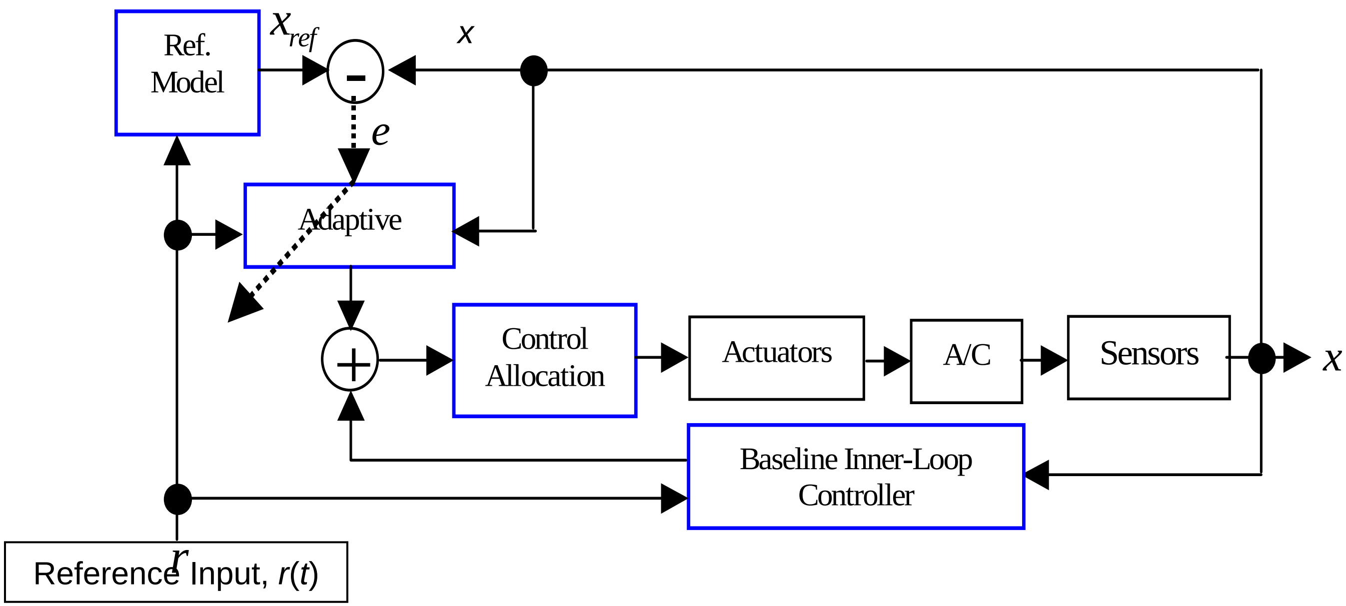
- Nominal Control: $u_{nom} = F_x^T\, x + F_r^T\, r$
- Adaptive Control: $u = \hat{K}_x^T\, x + \hat{K}_r^T\, r + \hat{\Theta}^T\, \Phi(x)$
- Augmentation:

$$
\begin{aligned}
u &= \hat{K}_x^T\, x + \hat{K}_r^T\, r + \hat{\Theta}^T\, \Phi(x) \pm u_{nom} \\
&= u_{nom} + \underbrace{\left(\hat{K}_x - F_x\right)}_{\hat{D}_x}{}^T x + \underbrace{\left(\hat{K}_r - F_r\right)}_{\hat{D}_r}{}^T r + \hat{\Theta}^T\, \Phi(x) \\
&= u_{nom} + \hat{D}_x^T\, x + \hat{D}_r^T\, x + \hat{\Theta}^T\, \Phi(x)
\end{aligned}
$$

- *Incremental* Adaptation:

$$
\begin{aligned}
\dot{\hat{D}}_x &= \Gamma_x \operatorname{Proj}\left(\hat{D}_x,\ \ -x\, e^T P\, B \operatorname{sgn}(\Lambda)\right), \quad \hat{D}_x = 0_{n\times m} \\
\dot{\hat{D}}_r &= \Gamma_r \operatorname{Proj}\left(\hat{D}_r,\ \ -r\, e^T P\, B \operatorname{sgn}(\Lambda)\right), \quad \hat{D}_r = 0_{m\times m} \\
\dot{\hat{\Theta}} &= \Gamma_\Theta \operatorname{Proj}\left(\hat{\Theta}, -\Phi(x)\, e^T P\, B \operatorname{sgn}(\Lambda)\right), \quad \hat{\Theta} = 0_{N\times m}
\end{aligned}
$$

Adaptive *Augmentation* Block-Diagram



- Reference Model provides desired response
- Nominal Baseline Controller
- Adaptive Augmentation
  - *Dead-Zone* modification prevents adaptation from changing nominal closed-loop dynamics
  - *Projection Operator* bounds adaptation parameters / gains

Adaptive Control using *Sigmoidal* NN

- System Dynamics: $\dot{x} = A x + B \Lambda \left(u - f(x) \right), \quad x \in R^n, \quad u \in R^m$
  - $A \in R^{n \times n}, \Lambda = \operatorname{diag}\left(\lambda_1 \quad \ldots \quad \lambda_m \right) \in R^{m \times m}$ are constant *unknown* matrices
  - $B \in R^{M \times m}$ is *known* constant matrix, and $M \geq m$
  - $\forall\, i = 1, \ldots, m \quad \operatorname{sgn}\left(\lambda_i \right)$ is *known*
- ***Approximation of uncertainty***:

$$f(x) = W^T \vec{\sigma}\left(V^T \mu \right) + \varepsilon_f(x), \quad \mu = \left(x^T \quad 1 \right)^T, \quad \varepsilon_f(x) \in R^m$$

  - *matrix* of constant *unknown* *Inner-Layer* weights: $V = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_N \\ \theta_1 & \ldots & \theta_N \end{bmatrix} \in R^{(n+1) \times N}$
  - *matrix* of constant *unknown* *Outer-Layer* weights: $W = \begin{bmatrix} \vec{w}_1 & \ldots & \vec{w}_m \\ c_1 & \ldots & c_m \end{bmatrix} \in R^{(N+1) \times m}$
  - vector of *N* *sigmoids* and a unity:

$$\vec{\sigma}\left(V^T \mu \right) = \left(\sigma\left(v_1^T x + \theta_1 \right) \quad \ldots \quad \sigma\left(\vec{v}_N^T x + \theta_N \right) \quad 1 \right)^T, \quad \text{where: } \sigma(s) = \frac{1}{1 + e^{-s}}$$

Adaptive Control using <u>*Sigmoidal*</u> NN

- Control Feedback: $u = \hat{K}_x^T x + \hat{K}_r^T r + \hat{W}^T \vec{\sigma}\left(\hat{V}^T \mu\right)$
  - (m n + m^2 + (n + 1) N + (N + 1) m) - parameters to estimate: $\hat{K}_x, \quad \hat{K}_r, \quad \hat{W}, \quad \hat{V}$
- Adaptation with Projection, $(\Lambda > 0)$:

$$\begin{cases} \dot{\hat{K}}_x = \Gamma_x \operatorname{Proj}\left(\hat{K}_x, -x e^T P B\right) \\ \dot{\hat{K}}_u = \Gamma_u \operatorname{Proj}\left(\hat{K}_u, -r e^T P B\right) \\ \dot{\hat{W}} = \Gamma_W \operatorname{Proj}\left(\hat{W}, \left(\vec{\sigma}\left(\hat{V}^T \mu\right) - \vec{\sigma}'\left(\hat{V}^T \mu\right)\hat{V}^T \mu\right) e^T P B\right) \\ \dot{\hat{V}} = \Gamma_V \operatorname{Proj}\left(\hat{V}, \mu e^T P B \hat{W}^T \vec{\sigma}'\left(\hat{V}^T \mu\right)\right) \end{cases}$$

- Provides <u>*bounded*</u> tracking

Design Example

Adaptive Reconfigurable Flight Control using RBF NN-s

Aircraft Model

- Flight Dynamics Approximation, (constant speed):

$$\dot{x}_p = A_p\, x_p + \underbrace{B\, G}_{B_p} \Lambda\left(\delta + K_0\left(x_p\right)\right) = A_p\, x_p + B_p\, \Lambda\left(\delta + K_0\left(x_p\right)\right)$$

  - State: $x_p = \begin{pmatrix}\alpha & \beta & p & q & r\end{pmatrix}^T$
  - Control allocation matrix *G*
  - *Virtual* Control Input: $\delta \in R^3$
  - Modeling control uncertainty / failures by $\Lambda \in R^{3\times 3}$ diagonal matrix with positive elements
  - Vector of actual control inputs:

    $G\,\Lambda\,\delta = \begin{pmatrix}\delta_{LOB} & \delta_{LMB} & \delta_{LIB} & \delta_{RIB} & \delta_{RMB} & \delta_{ROB} & \delta_{Tvec}\end{pmatrix}^T \in R^7$
  - A_p, B_p are *known* matrices
    - represent nominal system dynamics
  - *Matched* unknown nonlinear effects: $K_0\left(x_p\right) \in R^3$

Baseline Inner-Loop Controller

- Dynamics: $\dot{x}_c = A_c\, x_c + B_{1\,c}\, x_p + B_{2\,c}\, u$
- States: $x_c = \begin{pmatrix} q_I & p_I & r_I & r_w \end{pmatrix}^T \in R^4$
- Inner-loop commands, (reference input):

$$u = \begin{pmatrix} a_z^{cmd} & \beta^{cmd} & p^{cmd} & r^{cmd} \end{pmatrix}^T$$

- System output: $a_z = C_p\, x_p + \underbrace{D\,G}_{D_p}\Lambda\left(\delta + K_0\left(x_p\right)\right) = C_p\, x_p + D_p\,\Lambda\left(\delta + K_0\left(x_p\right)\right)$
- Augmented system dynamics:

$$\underbrace{\begin{pmatrix} \dot{x}_p \\ \dot{x}_c \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} A_p & 0 \\ B_{1c} & A_c \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_p \\ x_c \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} B_p \\ 0 \end{pmatrix}}_{B_1} \Lambda\left(\delta + K_0\left(x_p\right)\right) + \underbrace{\begin{pmatrix} 0 \\ B_{2c} \end{pmatrix}}_{B_2} u$$

$$\dot{x} = A\,x + B_1\,\Lambda\left(\delta + K_0\left(x_p\right)\right) + B_2\, u$$

- Inner-Loop Control: $\delta_L = K_x^T\, x + K_u^T\, u$

Reference Model

- Assuming nominal data, $\left(\Lambda = I_{3\times3}, \quad K_0\left(x_p\right) = 0_{3\times1}\right)$, and using baseline controller:

$$\dot{x}_{ref} = \underbrace{\left(A + B_1 K_x^T\right)}_{A_{ref}} x_{ref} + \underbrace{\left(B_2 + B_1 K_u^T\right)}_{B_{ref}} u = A_{ref}\, x_{ref} + B_{ref}\, u$$

- *__Assumption__*: Reference model matrix A_{ref} is Hurwitz, (i.e., baseline controller stabilizes nominal system)

Inner-Loop Control Objective
(Bounded Tracking)

- Design virtual control input such that, despite system uncertainties, the system state tracks the state of the reference model, while all closed-loop signals remain bounded
- Solution
  - Incremental, (i.e., adaptive augmentation), MRAC system with RBF NN, Dead-Zone, and Projection Operator

Adaptive Augmentation

- Total control input:

$$\delta = \underbrace{\hat{K}_x^T\, x + \hat{K}_u^T\, u - \hat{K}_0\left(x_p\right)}_{\text{Total Adaptive Control}} \pm \underbrace{\delta_L\left(x,u\right)}_{\text{Nominal Baseline}}$$

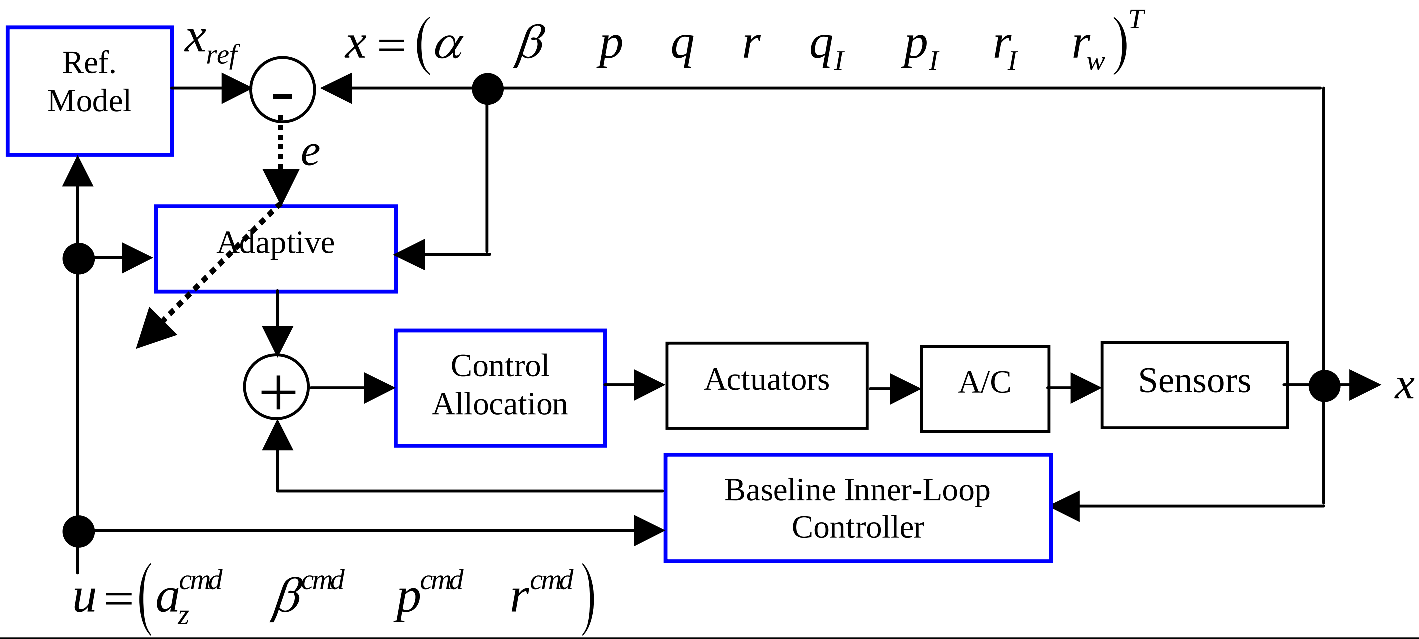
$$= \underbrace{\delta_L\left(x,u\right)}_{\delta_L(x,u)} + \underbrace{\left(\hat{K}_x - K_x\right)}_{\hat{k}_x}^T x + \underbrace{\left(\hat{K}_u - K_u\right)}_{\hat{k}_u}^T u - \underbrace{\hat{K}_0\left(x_p\right)}_{\hat{\Theta}^T\,\Phi\left(x_p\right)}$$

$$= \underbrace{\delta_L\left(x_p, x_c, u\right)}_{\text{Nominal Baseline}} + \underbrace{\Delta\hat{K}_x^T\, x + \Delta\hat{K}_u^T\, u - \hat{\Theta}^T\,\Phi\left(x_p\right)}_{\text{Incremental Adaptive Control}}$$

- Incremental adaptation with projection:

$$\begin{cases} \Delta\dot{\hat{K}}_x = \Gamma_x\ \mathrm{Proj}\left(\Delta\hat{K}_x,\ -x\, e^T\, P\, B_1\right), & \Delta\hat{K}_x\left(0\right) = 0_{n\times 3} \\ \Delta\dot{\hat{K}}_u = \Gamma_u\ \mathrm{Proj}\left(\Delta\hat{K}_u,\ -u\, e^T\, P\, B_1\right), & \Delta\hat{K}_u\left(0\right) = 0_{n\times 4} \\ \dot{\hat{\Theta}} = \Gamma_\Theta\ \mathrm{Proj}\left(\hat{\Theta},\ \Phi\left(x_p\right) e^T\, P\, B_1\right), & \hat{\Theta}\left(0\right) = 0_{N\times m} \end{cases}$$

Inner-Loop Block-Diagram



- Reference Model provides desired response
- Nominal Baseline Inner-Loop Controller
- Adaptive Augmentation
  - *Dead-Zone* modification prevents adaptation from changing nominal closed-loop dynamics
  - *Projection Operator* bounds adaptation parameters / gains

Adaptive Backstepping

Why?

- MRAC requires model matching conditions

$$A + B\Lambda K_x^T = A_m$$
$$B\Lambda K_r^T = B_m$$

- Example that violates matching
  - System: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b} u$
  - Reference model: $\begin{pmatrix} \dot{x}_1^m \\ \dot{x}_2^m \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}}_{A_m} \begin{pmatrix} x_1^m \\ x_2^m \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} r$

Matching conditions don't hold

$$A - A_m = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \neq b\, k_x^T = \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$$

Control Tracking Problem

- Consider 2nd order *cascaded* system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\, x_2$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)\, u$$

- Control goal
  - Choose u such that: $x_1(t) \to x_1^{com}(t), \quad \text{as } t \to \infty$
- Assumptions
  - All functions are known
  - $g_i \neq 0$ does not cross zero
- Example: AOA tracking

$$\begin{cases} \dot{\alpha} = \underbrace{-L_\alpha(\alpha)\overbrace{\alpha}^{x_1}}_{f_1} + \underbrace{1}_{g_1}\overbrace{q}^{x_2} \\ \dot{q} = \underbrace{M_0(\alpha, q)}_{f_2} + \underbrace{1}_{g_2}\underbrace{\dot{q}_{cmd}}_{u} \end{cases}$$

Backstepping Design

- Introduce pseudo control: $x_2^{com} = x_2^{com}(t)$
- Rewrite the 1st equation:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\,x_2^{com} + g_1(x_1)\underbrace{\left(x_2 - x_2^{com}\right)}_{\Delta x_2}$$

- Dynamic inversion using pseudo control:

$$x_2^{com} = \frac{1}{g_1(x_1)}\left(\dot{x}_1^{com} - f_1(x_1) - k_1\,\Delta x_1\right)$$

- 1st state error dynamics: $\Delta\dot{x}_1 = -k_1\,\Delta x_1 + g_1(x_1)\,\Delta x_2$

Backstepping Design (continued)

- Dynamic inversion using actual control

$$u = \frac{1}{g_2(x_1, x_2)}\left(\dot{x}_2^{com} - f_2(x_1, x_2) - k_2\,\Delta x_2 - g_1(x_1)\Delta x_1\right)$$

- 2nd state error dynamics

$$\Delta\dot{x}_2 = -k_2\,\Delta x_2 - g_1(x_1)\Delta x_1$$

- *Asymptotically stable* error dynamics

$$\begin{pmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{pmatrix} = \begin{pmatrix}-k_1 & g_1(x_1)\\ -g_1(x_1) & -k_2\end{pmatrix}\begin{pmatrix}\Delta x_1\\ \Delta x_2\end{pmatrix}$$

nonlinear system

- Conclusion: $x_i(t) \to x_i^{com}(t), \quad \text{as } t \to \infty$

Adaptive Backstepping Design

- 1st state dynamics: $\dot{x}_1 = \hat{f}_1 + \hat{g}_1\, x_2^{com} + \hat{g}_1\, \Delta x_2 - \Delta f_1 - \Delta g_1\, u$
  - Function estimation errors:

$$\Delta f_1 \triangleq \hat{f}_1 - f_1, \quad \Delta g_1 \triangleq \hat{g}_1 - g_1$$

- Dynamic inversion using pseudo control and estimated functions:

$$x_2^{com} = \frac{1}{\hat{g}_1(x_1)}\left(\dot{x}_1^{com} - \hat{f}_1(x_1) - k_1\, \Delta x_1\right)$$

- 1st state error dynamics:

$$\Delta\dot{x}_1 = -k_1\, \Delta x_1 + \hat{g}_1\, \Delta x_2 - \Delta f_1 - \Delta g_1\, u$$

Adaptive Backstepping Design (continued)

- 2nd state dynamics: $\dot{x}_2 = \hat{f}_2 + \hat{g}_2\, u - \Delta f_2 - \Delta g_2\, u$
  - Function estimation errors:

$$\Delta f_2 \triangleq \hat{f}_2 - f_2, \quad \Delta g_2 \triangleq \hat{g}_2 - g_2$$

- Dynamic inversion using actual control and estimated functions:

$$u = \frac{1}{\hat{g}_2\left(x_1, x_2\right)}\left(\dot{x}_2^{com} - \hat{f}_2\left(x_1, x_2\right) - k_2\, \Delta x_2 - \hat{g}_1\left(x_1\right) x_1\right)$$

- 2nd state error dynamics:

$$\Delta\dot{x}_2 = -k_2\, \Delta x_2 - \hat{g}_1\, \Delta x_1 - \Delta f_2 - \Delta g_2\, u$$

Adaptive Backstepping Design (continued)

- Combined error dynamics:

$$\begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_1 & \hat{g}_1(x_1) \\ -\hat{g}_1(x_1) & -k_2 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \begin{pmatrix} -\Delta f_1 - \Delta g_1 u \\ -\Delta f_2 - \Delta g_2 u \end{pmatrix}$$

- Uncertainty parameterization, function and parameter estimation errors:

$$\Delta f_i = \Delta\theta_{f_i}^T \Phi_f(x_1, x_2) - \varepsilon_{f_i}$$
$$\Delta g_i = \Delta\theta_{g_i}^T \Phi_g(x_1, x_2) - \varepsilon_{g_i}$$

$$\Delta\theta_{f_i} \triangleq \hat{\theta}_{f_i} - \theta_{f_i}$$
$$\Delta\theta_{g_i} \triangleq \hat{\theta}_{g_i} - \theta_{g_i}$$

Adaptive Backstepping Design (continued)

- Tracking error dynamics:

$$\underbrace{\begin{pmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{pmatrix}}_{\dot{e}} = \underbrace{\begin{pmatrix} -k_1 & \hat{g}_1 \\ -\hat{g}_1 & -k_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix}}_{e} - \underbrace{\begin{pmatrix} \Delta\theta_{f_1}^T & \Delta\theta_{g_1}^T \\ \Delta\theta_{f_2}^T & \Delta\theta_{g_2}^T \end{pmatrix}}_{\Delta\Theta^T} \underbrace{\begin{pmatrix} \Phi_f \\ \Phi_g\, u \end{pmatrix}}_{\Phi} + \underbrace{\begin{pmatrix} \varepsilon_{f_1} + \varepsilon_{g_1} u \\ \varepsilon_{f_2} + \varepsilon_{g_2} u \end{pmatrix}}_{\varepsilon}$$

$$\Updownarrow$$

$$\dot{e} = A e - \Delta\Theta^T \Phi + \varepsilon$$

- Stable robust adaptive laws:

$$\dot{\hat{\Theta}} = \Gamma\, \mathrm{Proj}\left(\hat{\Theta},\quad \Phi\, e^T\right)$$

- *Conclusion*: Bounded tracking

Adaptive Control in the Presence of Actuator Constraints*

*E. Lavretsky and N. Hovakimyan, “Positive μ – modification for stable adaptation in the presence of input constraints,” ACC, 2004.

Overview

- Problem: Assure stability of an adaptive control system in the presence of actuator position / rate saturation constraints.
- Solutions
  - Ad-hoc
  - Proof-by-simulation
  - Lyapunov based
    - limited class of systems
    - sufficient conditions are often hard-to-verify

not acceptable

preferred

- Need: *Theoretically justified* and *verifiable* conditions for stable adaptation and control design with a possibility of *avoiding* actuator saturation phenomenon.
- Design Solutions include modifications, (adaptive / fixed gain) to:
  - control input
  - tracking error
  - reference model

Known Design Solutions

- R. Monopoli, (1975)
  - adaptive modifications: tracking error and reference input
  - no theoretical stability proof
- S.P. Karason, A.M. Annaswamy, (1994)
  - adaptive modifications: reference input
  - rigorous stability proof
- E.N. Johnson, A.J. Calise, (2003)
  - pseudo control hedging (PCH)
    - fixed gain modification of reference input
- E. Lavretsky, N. Hovakimyan, (2004)
  - positive μ – modification
    - adaptive modification of control and reference inputs
    - rigorous stability proof and verifiable sufficient conditions
    - capability to completely avoid control saturation

Adaptive Control in the Presence of Input Constraints: Problem Formulation

battle damage

control failures

- System dynamics: $\dot{x}(t) = A\,x(t) + b\,\lambda\,u(t), \quad x \in R^n, u \in R$
  - *A* is *unknown* matrix, (emulates battle damage)
  - *b* is *known* control direction
  - $\lambda > 0$ is *unknown* positive constant, (control failures)
- Static actuator

amplitude saturation

$$u(t) = u_{\max}\,\mathrm{sat}\left(\frac{u_c}{u_{\max}}\right) = \begin{cases} u_c(t), & |u_c(t)| \le u_{\max} \\ u_{\max}\,\mathrm{sgn}\left(u_c(t)\right), & |u_c(t)| \ge u_{\max} \end{cases}$$

commanded input

- *Ideal* Reference model dynamics:

$$\dot{x}_m^*(t) = A_m\,x_m^*(t) + b_m\,r(t), \quad x_m^* \in R^n, r \in R$$

Hurwitz

bounded reference input

Preliminaries

safety zone

- Define: $u_{\max}^{\delta} = u_{\max} - \delta, \quad \text{where: } 0 < \delta < u_{\max}$
- Commanded control deficiency: → $\Delta u_c = u_{\max}^{\delta} \operatorname{sat}\left(\frac{u_c}{u_{\max}^{\delta}}\right) - u_c$
- Adaptive control with μ– modification, (*implicit* form):

$$u_c = \underbrace{k_x^T x + k_r r}_{u_{lin}} + \boxed{\mu \Delta u_c}$$

linear feedback / feedforward component

control deficiency feedback

- Need *explicit* form of u_c

Positive μ – modification

- Adaptive control with μ – mod is given by *convex combination* of u_{lin} and $u_{\max}^{\delta}\,\mathrm{sat}\left(\dfrac{u_{lin}}{u_{\max}^{\delta}}\right)$

$$u_c = \frac{1}{1+\mu}\left(u_{lin} + \mu\, u_{\max}^{\delta}\,\mathrm{sat}\left(\frac{u_{lin}}{u_{\max}^{\delta}}\right)\right) = \begin{cases} u_{lin}, & |u_{lin}| \le u_{\max}^{\delta} \\ \dfrac{1}{1+\mu}\left(u_{lin} + \mu\, u_{\max}^{\delta}\right), & u_{lin} > u_{\max}^{\delta} \\ \dfrac{1}{1+\mu}\left(u_{lin} - \mu\, u_{\max}^{\delta}\right), & u_{lin} < -u_{\max}^{\delta} \end{cases}$$

continuous in time but *not* continuously differentiable

$$\boxed{\delta = 0 \wedge (\mu = 0 \vee \mu = \infty)} \Rightarrow \boxed{u = u_{\max}\,\mathrm{sat}\left(\frac{u_{lin}}{u_{\max}}\right)}$$

Closed-Loop Dynamics

- μ – mod control: $u_c = u_{lin} + \mu \Delta u_c$
- System dynamics: $\dot{x} = Ax + b\lambda u_c + b\lambda \overbrace{(u - u_c)}^{\Delta u}$
- Closed-loop system:

linear control deficiency

$$\dot{x} = Ax + b\lambda u_{lin} + b\lambda \overbrace{(\mu \Delta u_c + \Delta u)}^{\Delta u_{lin} = u - u_{lin}}$$

$$\text{where: } \Delta u_{lin} = u_{\max} \operatorname{sat}\left(\frac{u_c}{u_{\max}}\right) - u_{lin}$$

↕

does not depend on μ explicitly

$$\dot{x} = \left(A + b\lambda k_x^T\right)x + b\lambda\left(k_r r + \Delta u_{lin}\right)$$

Adaptive Reference Model Modification

- Closed-loop system:

$$\dot{x} = \left(A + b\,\lambda\,k_x^T\right)x + b\,\lambda\left(k_r\,r + \Delta u_{lin}\right)$$

- Leads to consideration of ***adaptive*** reference model:

adaptive augmentation

$$\dot{x}_m = A_m\,x_m + b_m\left(r(t) + \boxed{k_u\,\Delta u_{lin}}\right), \quad \left|r(t)\right| \le r_{\max}$$

reference input

- Matching conditions:

$$\forall\,\lambda > 0\ \exists\left(k_x^* \in R^n, \quad k_r^* \in R, \quad k_u^* \in R\right)$$

$$\begin{cases} A + b\,\lambda\left(k_x^T\right)^* = A_m \\ \left.\begin{aligned} b\,\lambda\,k_r^* = b_m \\ b\,\lambda = b_m\,k_u^* \end{aligned}\right\} \Rightarrow k_u^*\,k_r^* = 1 \end{cases}$$

Adaptive Laws Derivation

- Tracking error: $e = x - x_m$
- Parameter errors: $\begin{cases} \Delta k_x = k_x - k_x^* \\ \Delta k_r = k_r - k_r^* \\ \Delta k_u = k_u - k_u^* \end{cases}$
- Tracking error dynamics:

$$\dot{e} = A_m\, e + b\,\lambda\left(\Delta k_x^T\, x + \Delta k_r\, r\right) - b_m\, \Delta k_u\, \Delta u_{lin}$$

- Lyapunov function:

$$V\left(e, \Delta k_x, \Delta k_r, \Delta k_u\right) = e^T\, P\, e + \lambda\left(\Delta k_x^T\, \Gamma_x^{-1}\, \Delta k_x + \gamma_r^{-1}\, \Delta k_r^2 + \gamma_u^{-1}\, \Delta k_u^2\right)$$

where: $P\, A_m + A_m\, P = -Q < 0$

Stable Parameter Adaptation

- Adaptive laws derived to yield stability:

$$\begin{aligned}\dot{k}_x &= -\Gamma_x \, x\, e^T P\, b \\ \dot{k}_r &= -\gamma_r \, r(t) e^T P\, b \\ \dot{k}_u &= \gamma_u \, \Delta u_{lin} \, e^T \, P\, b_m\end{aligned} \Leftrightarrow \dot{V} = -e^T Q\, e < 0 \Rightarrow \dot{V}\left(e, \Delta k_x, \Delta k_r, \Delta k_u\right) \le 0$$

- For open-loop stable systems – global result
- For open-loop unstable systems verifiable sufficient conditions established:
  - upper bound on $r_{\max}$
  - lower bound on μ
  - upper bounds on initial conditions *x*(0) and Lyapunov function *V*(0)

$\mu -$ mod Design Steps

- Choose "safety zone" $0 < \delta < u_{\max}$ and sufficiently large $\mu > 0$
- Define *virtual* constraint: $u_{\max}^{\delta} = u_{\max} - \delta$
- *Linear* component of adaptive control signal: $u_{lin} = k_x^T x + k_r r(t)$
- *Total* adaptive control with $\mu -$ mod:

$$u_c = \frac{1}{1+\mu}\left(u_{lin} + \mu\, u_{\max}^{\delta}\, \mathrm{sat}\left(\frac{u_{lin}}{u_{\max}^{\delta}} \right) \right)$$

adaptive laws

$$\begin{cases} \dot{k}_x = -\Gamma_x\, x\, e^T P\, b \\ \dot{k}_r = -\gamma_r\, r(t)\, e^T P\, b \\ \dot{k}_u = \gamma_u\, \Delta u_{lin}\, e^T\, P\, b_m \end{cases}$$

$$\dot{x}_m = A_m\, x_m + b_m \left(r + k_u \underbrace{\left(u_{\max}\, \mathrm{sat}\left(\frac{u_c}{u_{\max}} \right) - u_{lin} \right)}_{\Delta u_{lin}} \right)$$

modified reference model

Simulation Example

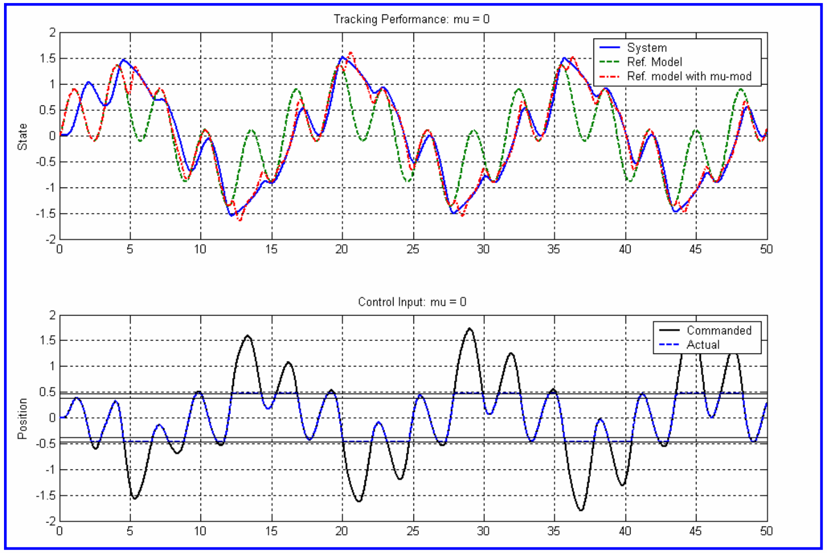
- Unstable open-loop system:

$$\dot{x} = a\,x + b\,u_{\max}\,\text{sat}\left(\frac{u_c}{u_{\max}}\right), \text{ where: } a = 0.5,\ b = 2,\ u_{\max} = 0.47$$

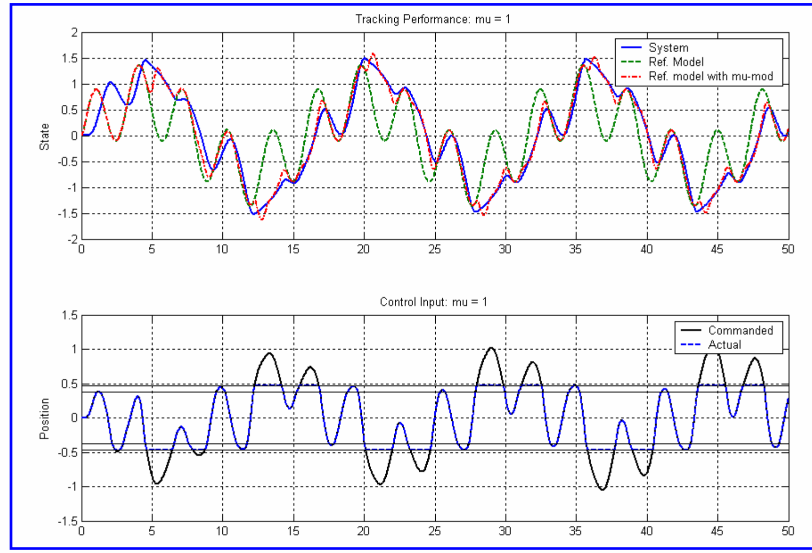
- Choose: $\delta = 0.2\,u_{\max}$ → $u_{\max}^{\delta} = u_{\max} - \delta = 0.8\,u_{\max}$
- Ideal reference model: $\dot{x}_m = -6\left(x_m - r(t)\right)$
- Reference input: $r(t) = 0.7\left(\sin(2t) + \sin(0.4t)\right)$
- Adaptation rates set to unity
- System and reference model start at zero

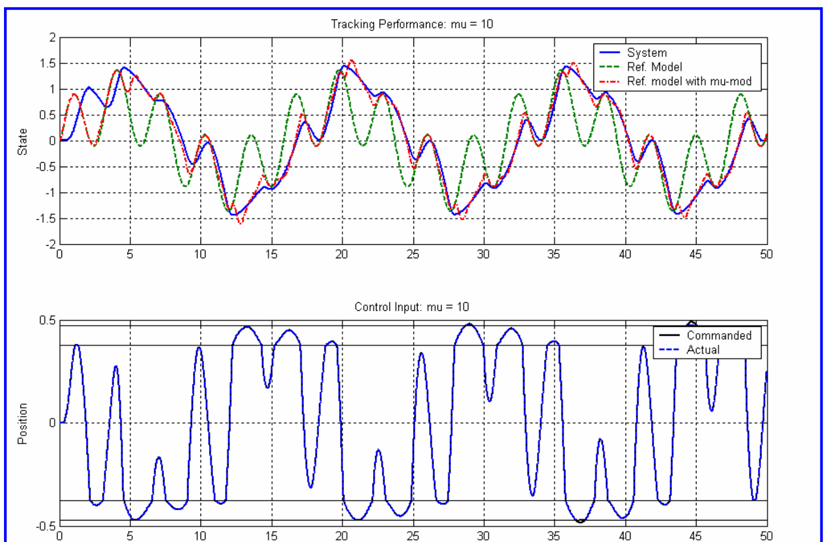
Simulation Data

$\mu = 0$

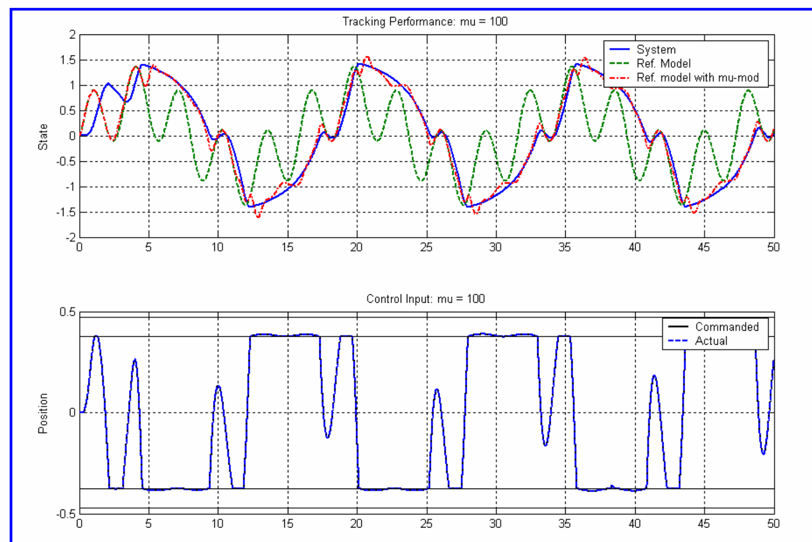


$\mu = 1$





$\mu = 10$



$\mu = 100$

μ – mod Design Summary

- Lyapunov based
- Provides closed-loop stability and bounded tracking
  - convex combination of linear adaptive control and its $u_{\max}^{\delta}$ – limited value
  - adaptive reference model modification
- Verifiable sufficient conditions
- **Future Work**
  - MIMO systems
  - Dynamic actuators
  - Nonaffine-in-control dynamics
  - Flight control applications

Adaptive Flight Control Applications, Open Problems, and Future Work

Autonomous Formation Flight, (AFF)

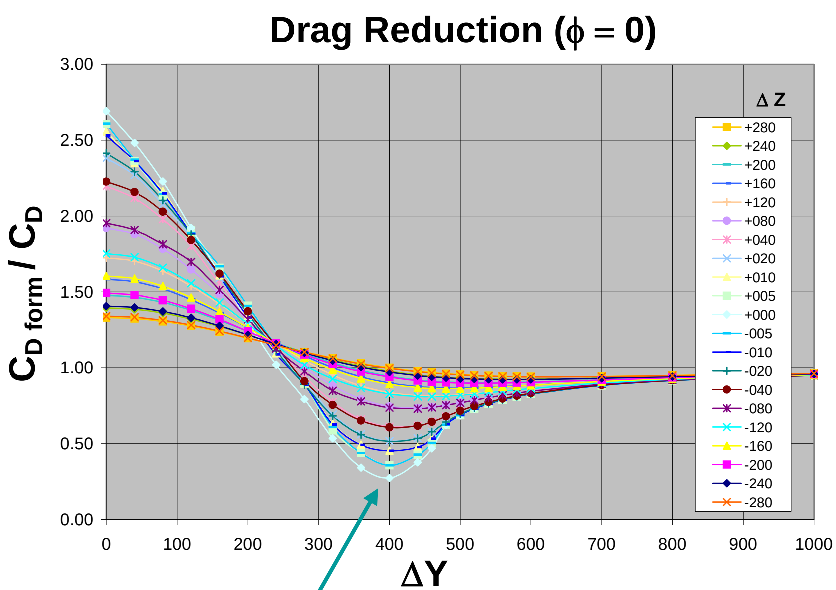
References:

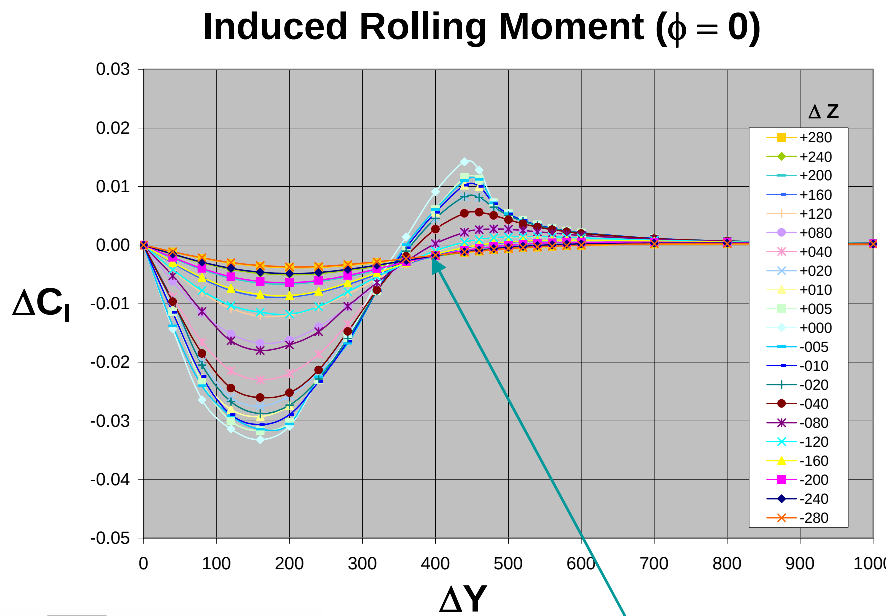
- Lavretsky, E. "F/A-18 Autonomous Formation Flight Control System Design", *AIAA GN&C Conference, Monterey, CA, 2002.*
- Lavretsky, E., Hovakimyan, N., Calise, A., Stepanyan, V. *"Adaptive Vortex Seeking Formation Flight Neurocontrol", AIAA-2002-4757*, *AIAA GN&C Conference,St. Antonio, TX, 2003.*

AFF: Program Overview

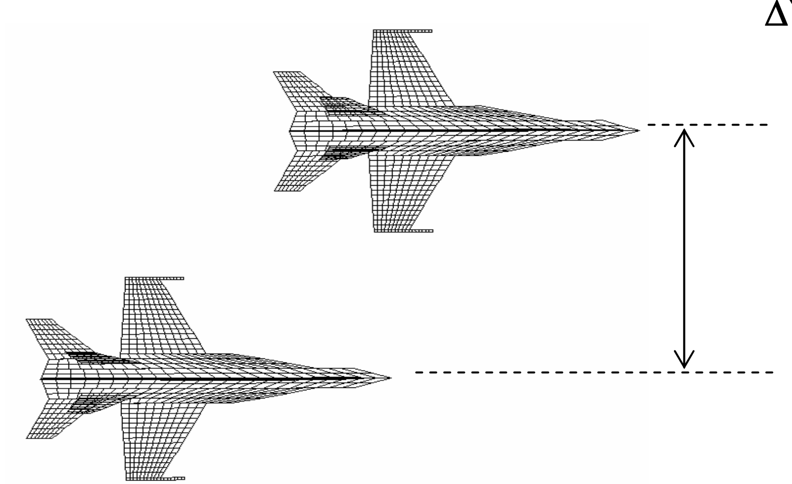
- **Program participants**:
  - NASA Dryden
  - Boeing - Phantom Works
  - UCLA
- **Flight test program**
  - Completed in December of 2001
  - 2 F/A-18 Hornets, 45 flights
  - Demonstrated up to 20% induced aerodynamic drag reduction
- **AFF Autopilot**
  - **Baseline** linear classical design to meet stability margins
  - **Adaptive** incremental system to counteract unknown vortex effects and environmental disturbances
  - **On-line extremum seeking** command generation

AFF: Lead Aircraft Wingtip Vortex Effects
Induced Drag Ratio & Rolling Moment Coefficient





“sweet” spot



close to vortex induced roll reversal

AFF: Trailing Aircraft Dynamics in Formation

- Trailing Aircraft:

Longitudinal Dynamics

$$\begin{cases} \dot{l} = V \\ \dot{V} = -g\sin\gamma + \dfrac{\rho V^2}{2m} S\left(C_{T_{\delta_T}}\delta_T - C_D(M,\alpha)\,\eta(y,\phi)\right) \end{cases}$$

Throttle Input

Lateral Dynamics

$$\begin{cases} \ddot{y} = -g\, n_z \sin\phi \\ \dot{\phi} = p \\ \dot{p} = -a_p\, p + b_{\delta_a}\delta_a + \xi(y,\phi) \end{cases}$$

Aileron Input

- Trailing Aircraft Modeling Assumptions
  - SCAS yields 1st order roll dynamics & turn coordination
  - $a_p, b_{\delta_a}, C_{T_{\delta_T}}$ are *unknown positive* constants
  - $C_D(M,\alpha),\ \eta(y,\phi),\ \xi(y,\phi)$ are *unknown bounded* functions of known arguments and shapes
- Lead aircraft trimmed for level flight

AFF: Vortex Seeking Formation Flight Control

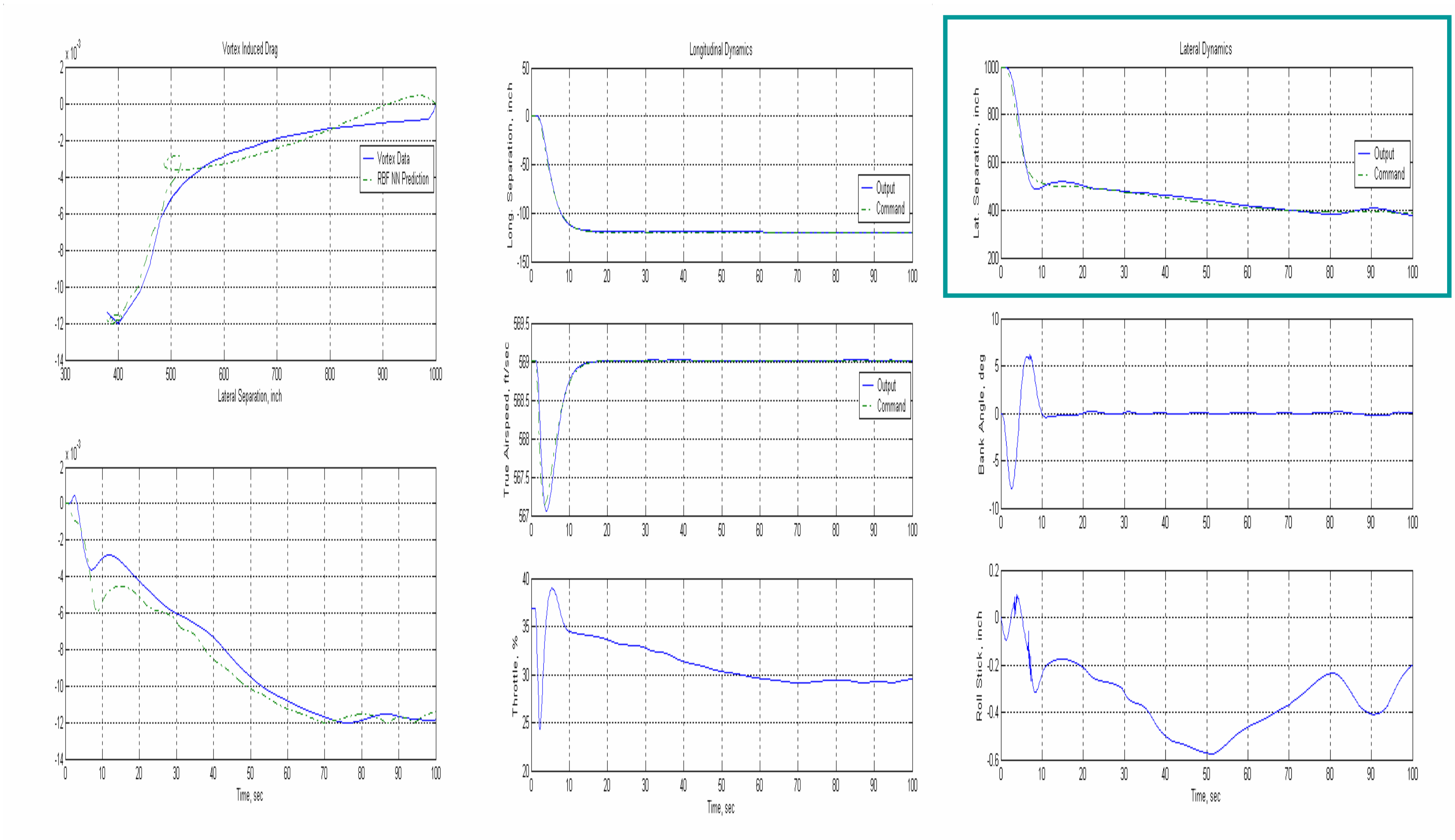
- **Problem**: Using *throttle* and *aileron* inputs
  - Track desired longitudinal displacement command l_c
  - Generate on-line and track lateral separation command y_c in order to:
    - Minimize unknown vortex induced drag coefficient $\eta(y,\phi)$ with respect to its 1st argument, (lateral separation)

$$\dot{V} = -g\sin\gamma + \frac{\rho V^2}{2m} S\left(C_{T_{\delta_T}}\,\delta_T - C_D(M,\alpha)\,\eta(y,\phi)\right)$$

- Remarks:
  - Aileron controls lateral separation
  - Throttle controls longitudinal separation
    - depends on lateral separation through unknown function $\eta(y,\phi)$
- **Solution**
  - Using Direct Adaptive Model Reference Control
  - Radial Basis Functions for approximation of uncertainties
  - Extremum Seeking Command Generation

$$\dot{y}_r = -\gamma \left.\frac{\partial\hat{\eta}(y,\phi)}{\partial y}\right|_{y=y_r}, \quad \gamma > 0$$

AFF: Simulation Data

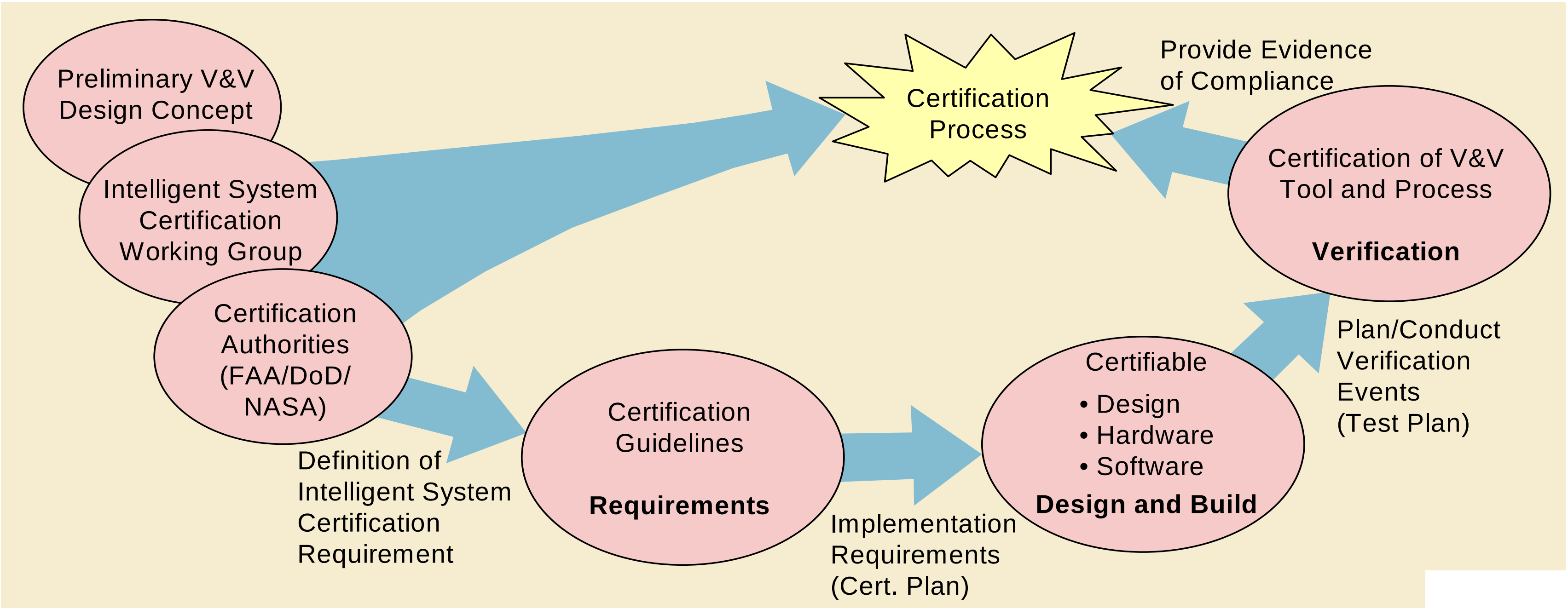


Open Problems and Future Work

Task 1: Validation & Verification (V&V) of Adaptive Systems

- Significant industry effort going into development of adaptive / reconfigurable GN&C systems
- Methods to test and certify flight critical systems are not readily available
- There exists a necessity to develop V&V methods and certification tools that are similar to and extend the current process for conventional, non-adaptive GN&C systems
- Theoretically justified V&V technologies are needed to:
  - provide a standard process against which adaptive GN&C systems can be certified
  - offer certification guidelines during the early design cycle of such systems

Task 1: V&V of Adaptive Systems
Road Map to Solution (Issue Paper)



- **Goal:** Provide *theoretically* justified V&V method and a process-based acceptance procedure to certify current and future intelligent / adaptive GN&C flight critical systems

- **Two Major Tasks**
  - Stability Margins / Robustness Analysis
  - S/W V&V Procedures

Task 1: V&V of Adaptive Systems

Subtask: Theoretical Stability / Robustness Analysis

- Establish adaptive control design guidelines
  - Define rates of adaptation
  - Calculate stability / robustness margins
  - Determine bounds on control parameters that correspond to stability / robustness margins
- Perform system validation using the derived margins
- Incorporate modifications that lead to improvement (if required) in the stability / robustness margins
- Validate closed-loop system tracking performance

Task 2: Integrated Vehicle Health Management (IVHM) and Composite Adaptation

- Aerodynamic parameters are of paramount importance to IVHM system functionality
- Examine different sources of on-line aerodynamic parameter estimation
  - Tracking errors
  - Prediction errors
- Composite Adaptive Flight Control = (Indirect + Direct) MRAC

Task 3: Persistency of Excitation in Flight Mechanics

- Information content from adaptation / estimation processes depends on parameter convergence
  - Requires persistent excitation (PE) of control inputs
- Need numerically stable / on-line verifiable PE conditions for flight mechanics and control
- *Aircraft Example*: Longitudinal dynamics

$$\begin{cases} \dot{V} = \dfrac{T\cos\alpha - D}{m} - g\sin(\theta-\alpha) \\ \dot{\alpha} = q - \dfrac{T\sin\alpha + L}{mV} + g\cos(\theta-\alpha) \\ \dot{q} = \dfrac{M}{I_y} \\ \dot{\theta} = q \end{cases}$$

$$\begin{cases} T = \bar{q}\,S\,C_T \cong \bar{q}\,S\,C_{T_{\delta_T}}\,\delta_T \\ L = \bar{q}\,S\,C_L \cong \bar{q}\,S\,C_L(\alpha,q) \\ D = \bar{q}\,S\,C_D \cong \bar{q}\,S\,C_D(\alpha,q) \\ M = \bar{q}\,S\,\bar{c}\,C_M \cong \bar{q}\,S\,\bar{c}\left(C_{M_0}(\alpha,q) + C_{M_{\delta_e}}(\alpha,\delta_e)\,\delta_e\right) \end{cases}$$

Control Inputs (δ_T, δ_e)

Problem:

➢Estimate on-line unknown aerodynamic coefficients

➢Find sufficient conditions (PE) that yield convergence of the estimated parameters to their corresponding true (unknown) values

Design Example: F-16 Adaptive Pitch Rate Tracker

Aircraft Data
Short-Period Dynamics

- **Trim conditions**
  - CG = 35%, Alt = 0 ft, QBAR = 300 psf, V_T = 502 fps, AOA = 2.1 deg
- **Nominal system**
  - statically unstable
  - open-loop dynamically stable, (2 real negative eigenvalues)
- **Control architecture**
  - baseline / nominal controller
    - LQR pitch tracking design
  - direct adaptive model following augmentation
- **Simulated failures**
  - elevator control effectiveness: 50% reduction
  - battle damage instability
    - static instability: 150% increase
    - pitch damping: 80% reduction
  - pitching moment modeling nonlinear uncertainty

LQR PI Baseline Controller

- Using LQR PI state feedback design
  - nominal values for stability & control derivatives
  - pitch rate step-input command
  - no uncertainties, no control failures
  - system dynamics:
  - "wiggle" system in matrix form

$$\begin{cases} \dot{e}_q^I = q - q^{cmd} \\ \dot{\alpha} = \dfrac{Z_\alpha}{V}\alpha + q + \dfrac{Z_\delta}{V}\delta_e \\ \dot{q} = M_\alpha \alpha + M_q q + M_\delta \delta_e \end{cases}$$

$$\underbrace{\begin{pmatrix} \dot{e}_q \\ \ddot{\alpha} \\ \ddot{q} \end{pmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & \dfrac{Z_\alpha}{V} & 1 \\ 0 & M_\alpha & M_q \end{pmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} e_q \\ \dot{\alpha} \\ \dot{q} \end{pmatrix}}_{\tilde{x}} + \underbrace{\begin{pmatrix} 0 \\ \dfrac{Z_\delta}{V} \\ M_\delta \end{pmatrix}}_{\tilde{B}} \underbrace{\dot{\delta}_e}_{\tilde{u}} \quad \Leftrightarrow \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}$$

LQR PI Baseline Controller (continued)

- LQR design for the "wiggle" system
  - Optimal feedback solution: $\tilde{u} = -\tilde{K}\,\tilde{x}$
  - Using original states:

$$\dot{\delta}_e^{bl} = -\begin{pmatrix} K_q^I & K_\alpha & K_q \end{pmatrix} \begin{pmatrix} e_q \\ \dot{\alpha} \\ \dot{q} \end{pmatrix} = -K_q^I\, e_q - K_\alpha\, \dot{\alpha} - K_q\, \dot{q}$$

  - Integration yields LQR PI feedback: $\delta_e^{bl} = -\overbrace{K_x}^{\tilde{K}}\, x$

$$\delta_e^{bl} = -K_q^I\, e_q^I - K_\alpha\, \alpha - K_q\, q$$

$$\delta_e^{bl} = -10\, e_q^I - 3.2433\,\alpha - 10.7432\, q$$

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1.0189 & 0.9051 \\ 0 & 0.8223 & -1.0774 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 \\ -0.0022 \\ -0.1756 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 100 \end{pmatrix}$$

closed-loop eigenvalues

Eigenvalue	Damping	Freq. (rad/s)
-7.97e-001 + 3.45e-001i	9.18e-001	8.68e-001
-7.97e-001 - 3.45e-001i	9.18e-001	8.68e-001
-2.39e+000	1.00e+000	2.39e+000

Short-Period Dynamics with Uncertainties

- **System:**

$$\underbrace{\begin{pmatrix} \dot{e}_q^I \\ \dot{\alpha} \\ \dot{q} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{Z_\alpha}{V} & 1 \\ 0 & M_\alpha & M_q \end{pmatrix}}_{\tilde{A}} \underbrace{\begin{pmatrix} e_q^I \\ \alpha \\ q \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ \frac{Z_\delta}{V} \\ M_\delta \end{pmatrix}}_{B_1 = \tilde{B}} \Lambda\left(\delta_e + K_0(\alpha, q)\right) + \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}}_{B_2} q^{cmd}$$

- **Reference model:**

$$\dot{x} = A\,x + B_1\,\Lambda\left(\delta_e + K_0(\alpha, q)\right) + B_2\, q^{cmd}$$

  - no uncertainties
  - (Plant + Baseline LQR PI)

$$\dot{x}_{ref} = \underbrace{\left(A + B_1 K_x^T\right)}_{A_{ref}} x_{ref} + \underbrace{B_2}_{B_{ref}} q^{cmd} = A_{ref}\, x_{ref} + B_{ref}\, q^{cmd}$$

- **Control Goal**
  - Model following pitch rate tracking: $\left\| x - x_{ref} \right\| \to 0$ (tracking error vector)

Adaptive Augmentation Design

- Total elevator deflection:

$$\delta_e = \delta_e^{bl} + \delta_e^{ad} = \underbrace{K_q^I\, e_q^I + K_\alpha\, \alpha + K_q\, q}_{\delta_e^{bl}} + \underbrace{\hat{k}_q^I\, e_q^I + \hat{k}_\alpha\, \alpha + \hat{k}_q\, q - \overbrace{\hat{\Theta}^T\, \Phi(\alpha, q)}^{\hat{K}_0(\alpha,q)}}_{\delta_e^{ad}}$$

$$\Updownarrow$$

$$\delta_e = \left(K_x + \hat{k}_x\right)^T x - \hat{\Theta}^T\, \Phi(\alpha, q)$$

- Adaptive laws:

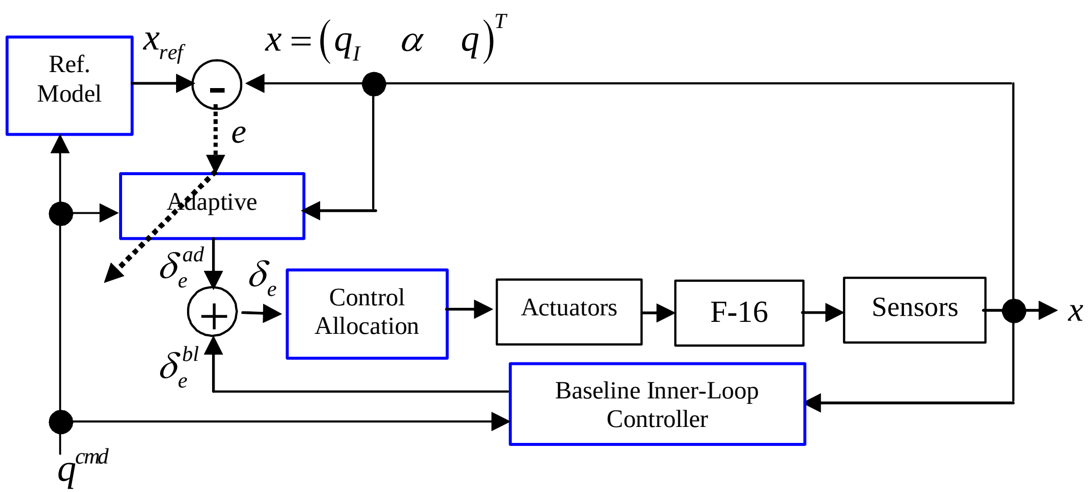
$$\begin{cases} \dot{\hat{k}}_x = \Gamma_x \operatorname{Proj}\left(\hat{k}_x,\, -x\, e^T\, P\, B_1\right) \\ \dot{\hat{\Theta}} = \Gamma_\Theta \operatorname{Proj}\left(\hat{\Theta},\, \Phi\left(x_p\right) e^T\, P\, B_1\right) \end{cases} \Leftrightarrow \begin{cases} \begin{pmatrix} \dot{\hat{k}}_\alpha \\ \dot{\hat{k}}_q \\ \dot{\hat{k}}_q^I \end{pmatrix} = \Gamma_x \operatorname{Proj}\left(\begin{pmatrix} \hat{k}_\alpha \\ \hat{k}_q \\ \hat{k}_q^I \end{pmatrix}, -\begin{pmatrix} \alpha \\ q \\ q_I \end{pmatrix}\begin{pmatrix} q_I - q_I^{ref} & \alpha - \alpha_{ref} & q - q_{ref} \end{pmatrix} P \begin{pmatrix} 0 \\ \frac{Z_\delta}{V} \\ M_\delta \end{pmatrix}\right) \\ \dot{\hat{\Theta}} = \Gamma_\Theta \operatorname{Proj}\left(\hat{\Theta},\, \Phi(\alpha, q)\begin{pmatrix} q_I - q_I^{ref} & \alpha - \alpha_{ref} & q - q_{ref} \end{pmatrix} P \begin{pmatrix} 0 \\ \frac{Z_\delta}{V} \\ M_\delta \end{pmatrix}\right) \end{cases}$$

Adaptive Augmentation Design (continued)

- **Free design parameters**
  - symmetric positive definite matrices: $\left(Q,\ \Gamma_x,\ \Gamma_\Theta\right)$
- **Need to solve algebraic Lyapunov equation**

$$P\,A_{ref} + A_{ref}^T\,P = -Q$$

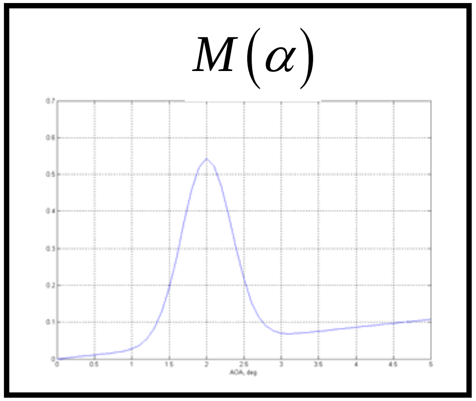
- Using **Dead-Zone** modification and **Projection Operator**



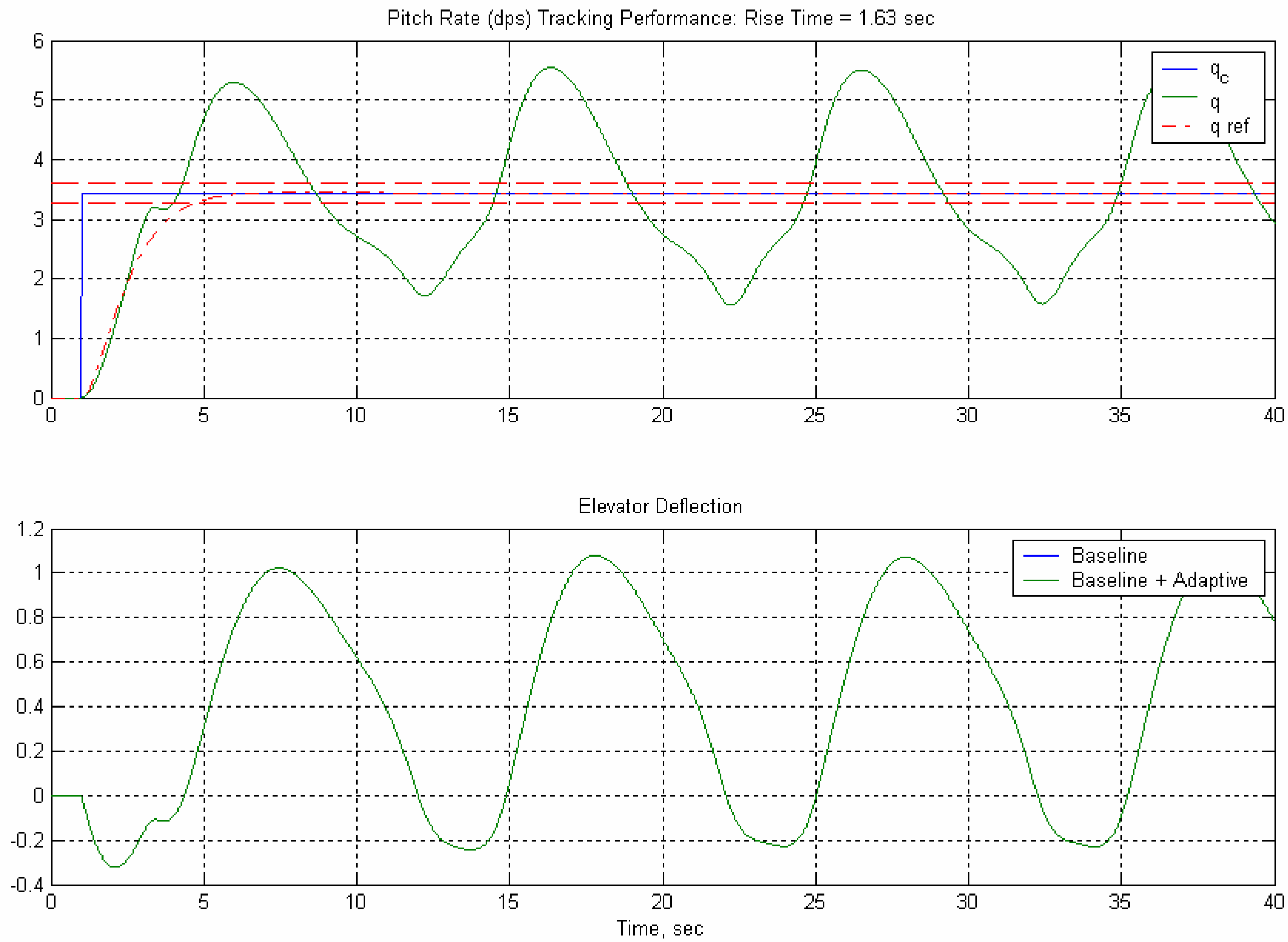
Adaptive Design Data

- **Design parameters**
  - using 11 RBF functions: $\phi_i = e^{-\frac{(\alpha-\alpha_i)^2}{\sigma^2}}, \quad \alpha_i \in [-10:.1:10]$
  - Rates of adaptation:
    $\Gamma_x = 0, \quad \Gamma_\Theta = 1$
  - Solving Lyapunov equation with: $Q = \mathrm{diag}([0 \quad 1 \quad 800])$
- **Zero initial conditions**
- **Pitch rate command input**
- **System Uncertainties**
  - 50% elevator effectiveness failure, $(0.5 * M_\delta^{bl})$
  - 50% increase in static instability, $(1.5 * M_\alpha^{bl})$
  - 80% decrease in pitch damping, $(0.2 * M_q^{bl})$
  - nonlinear pitching moment

$$M(\alpha) = 1.5 * M_\alpha^{bl} + e^{-\frac{\left(\alpha - \frac{2\pi}{180}\right)^2}{0.0116^2}}$$

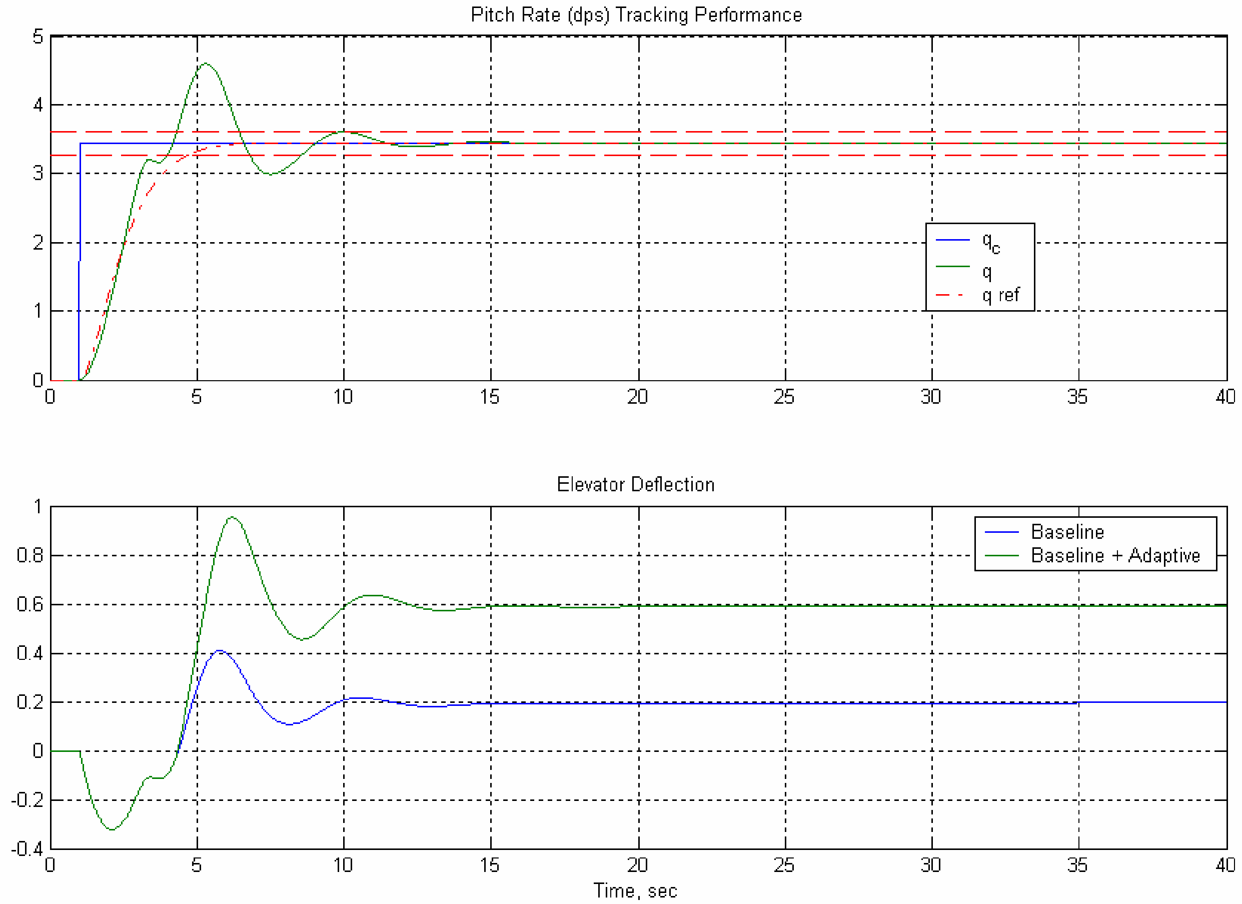


LQR PI: Tracking Step-Input Command



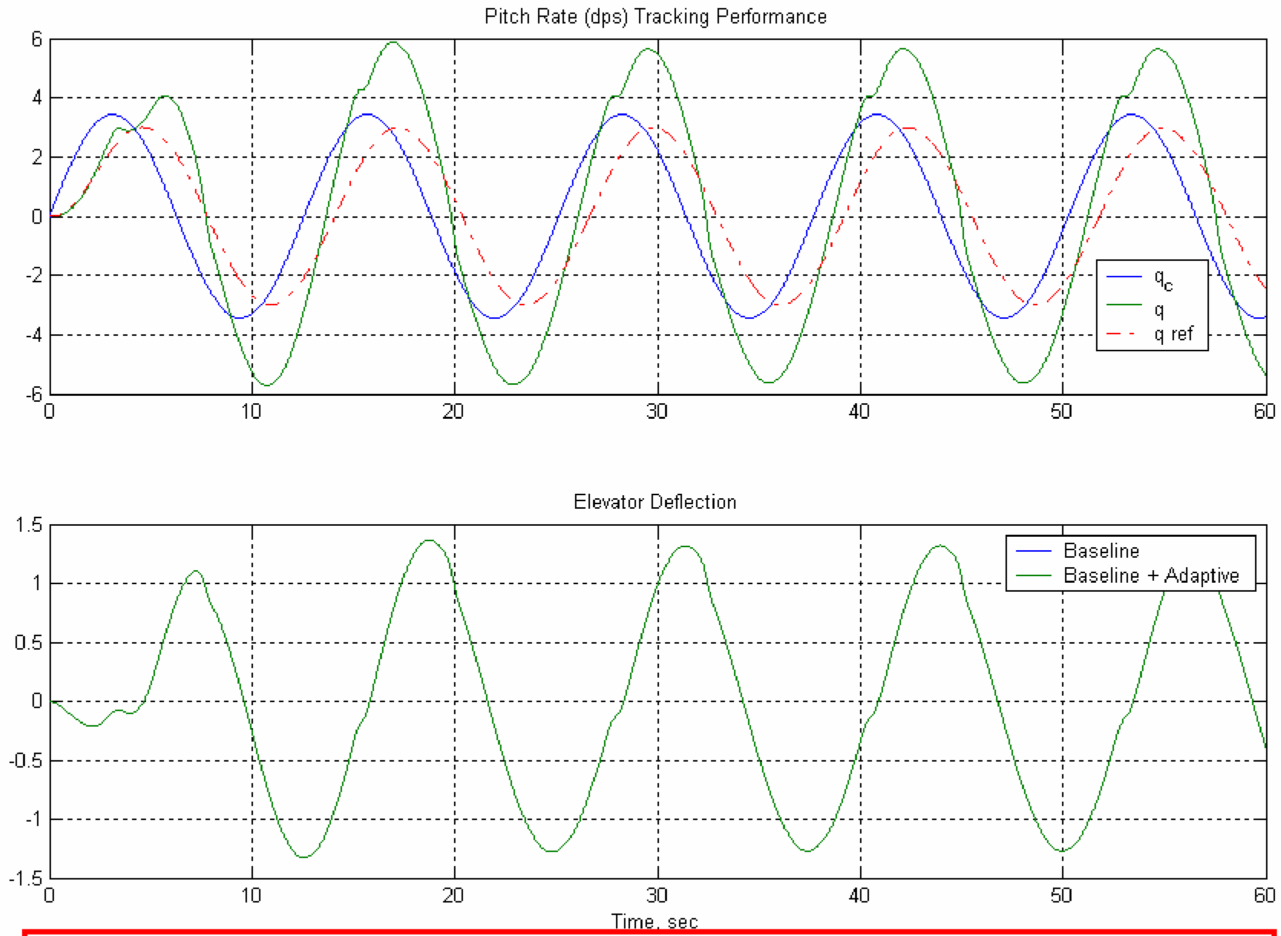
Unstable Dynamics due to Uncertainties

LQR PI + Adaptive: Tracking Step-Input Command



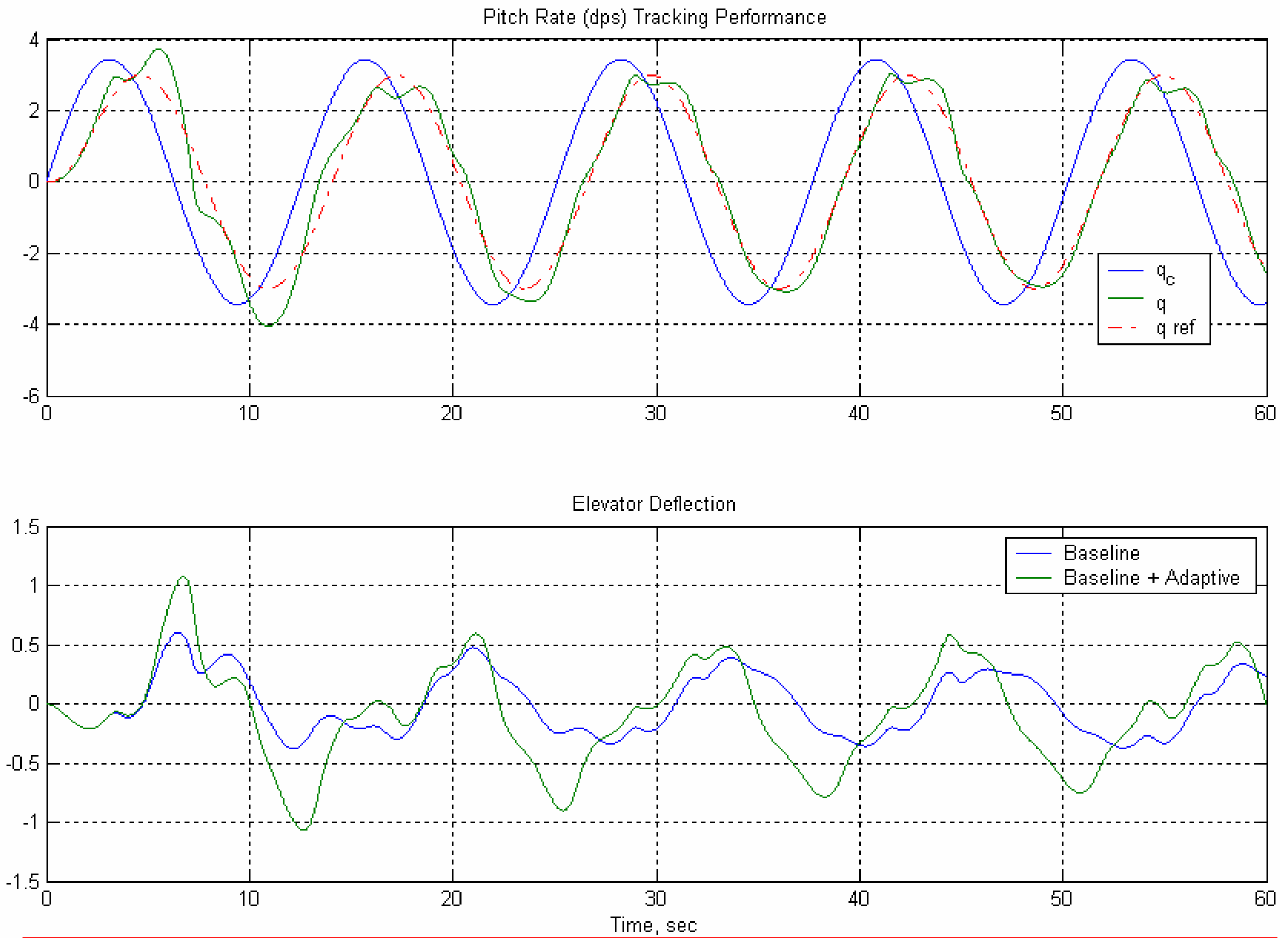
Adaptive Augmentation yields Bounded Stable Tracking in the Presence of Uncertainties

LQR PI: Tracking Sinusoidal Input with Uncertainties



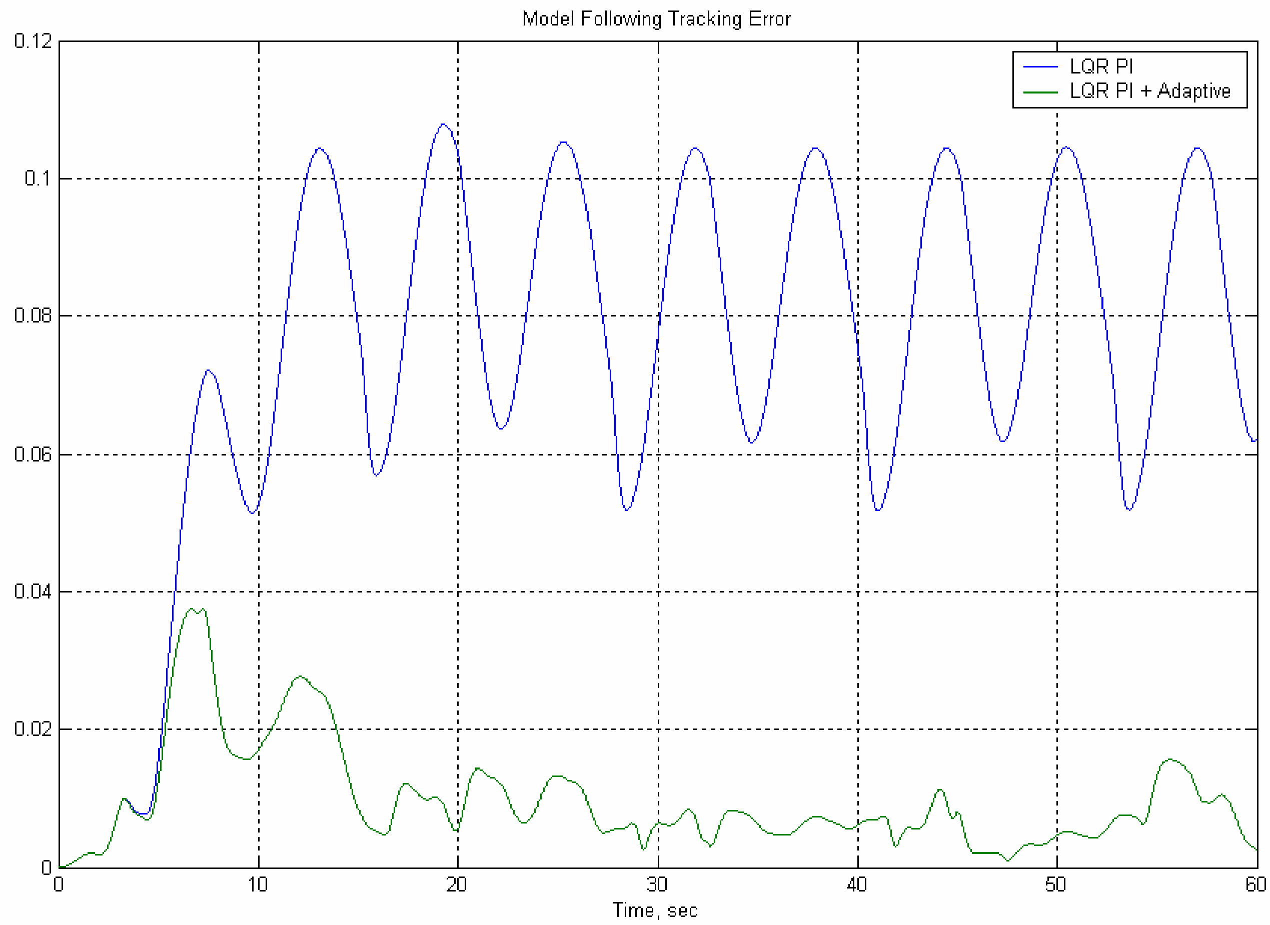
LQR PI Tracking Performance Degradation in the Presence of Uncertainties

LQR PI + Adaptive: Tracking Sinusoidal Input with Uncertainties



Adaptive Augmentation Recovers Target Tracking Dynamics in the Presence of Uncertainties

Model Following Tracking Error Comparison



Adaptive Augmentation yields Significant Reduction in Tracking Error Magnitude

Adaptive Design Comments

- **RBF NN adaptation dynamics**

$$\dot{\hat{\Theta}}_i = \left(\Gamma_\Theta\right)_{ii} \Phi_i\left(\alpha, q\right)\left(k_{1i}\left(q_I - q_I^{ref}\right) + k_{2i}\left(\alpha - \alpha_{ref}\right) + k_{3i}\left(q - q_{ref}\right)\right)$$

- **Fixed RBF NN gains**
  - simulation data

$$k_{1i} = 0, \quad k_{2i} = -1.1266, \quad k_{3i} = -24.0516$$

$$\begin{pmatrix} k_{1i} \\ k_{2i} \\ k_{3i} \end{pmatrix} = P \begin{pmatrix} 0 \\ \frac{Z_\delta}{V} \\ M_\delta \end{pmatrix} \left(\Gamma_\Theta\right)_{ii}$$

- **Projection Operator**
  - keeps parameters bounded
  - nonlinear extension of anti-windup integrator logic
- **Dead-Zone modification**
  - freezes adaptation process if: $\left\| x - x_{ref} \right\| \le \varepsilon$ (dead-zone tolerance)
  - separates adaptive augmentation from baseline controller